1829

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De

POTESTATIBUS SINUUM ET COSINUUM

QUAE SECUNDUM SINUS AUT COSINUS MULTIPLICIUM ARCUUM PROCEDUNT

DISSERTATIO

QUAM

AMPLISSIMI ORDINIS PHILOSOPHORUM VRATISLAVIENSIS AUCTORITATE

PRO

SUMMIS IN PHILOSOPHIA HONORIBUS RITE OBTINENDIS

D. XI. JULII a. MDCCCXXIX

H. L. Q. C.

PUBLICE DEFENDET

AUCTOR

FRANCISCUS BREDOW
COLLEGA GYMNASII OLSNENSIS.

OLSNAE, MDCCCXXIX.
TYPIS DESCRIPSIT LUDWIG.

Theses.

I. Series, quae sinum et cosinum per arcuum potestates exprimunt, nempe

$$\Sigma \frac{(-1)^{a} \varphi^{2a+1}}{\Pi 2a+1} = \text{Sin. } \varphi$$

$$\Sigma \frac{(-1)^{a} \varphi^{2a}}{\Pi 2a} = \text{Cos. } \varphi$$

modo valent, si φ arcum inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ significat.

II. $x^{\frac{3}{15}}$ non poni potest $= x^{\frac{1}{5}}$.

III. Ad phaenomena a calore effecta explicanda non necessarium est, ut materia quaedam fingatur.

IV. Phidias Jovem Olympium ante majora opera Athenis statuta perfecit Ol. 83. Mortuus est Athenis Ol. 86, 1.

VIRO ERUDITISSIMO

ERNESTO PLAMANN

PHILOSOPHIAE DOCTORI PROFESSORI BEROLINENSI

HAS

STUDIORUM PRIMITIAS

GRATI ANIMI TESTES ESSE

VOLUIT

AUCTOR.

Potestates sinuum et cosinuum, quae secundum sinus aut cosinus multiplicium arcuum procedunt, explicatae.

§. 1.

Illustris Leonhardus Euler primus huic quaestioni solvendae operam navavit et quod ille effecit, eo magis ab omnibus pro certo sumebatur, quia postea Lagrange ratione prorsus discrepante ad eandem summam pervenit. Fortuito autem Poisson, quum secundum Euleri formulam exemplum quoddam ad calculos vocaret, summamque omnino falsam consequeretur, illam solutionem non rectam esse animadvertit. Quo facto eadem res a permultis praecipue Francogallis et Germanis tractabatur et occasione data serierum et potestatum explicandarum principia subtilius exquirebantur, ut eluceret, quomodo viri, acumine ingenii florentes, per errorem labi potuerint. Quum autem illi hac in re versantes non omnes idem, sed varia invenerint, non superfluum videtur, quae de illa re litteris tradita sunt, componere et demonstrare, quae vera sit summa et quomodo variae ejus expressiones inter se consentiant.

§. 2.

De significationibus adhibendis. — In binomialibus coefficientibus significandis Besselii rationem sequor, supra P numerum termini notans et infra potestatem, ad quam radix eruenda est; itaque

$$P_m^{\mathfrak{a}} = \frac{m.m-1.m-2\ \ldots\ (m-\mathfrak{a}+1)}{1.\ 2.\ 3\ \ldots\ \mathfrak{a}}$$

quia termini a secundo totius seriei numerandi sunt. Primi autem termini coefficiens est $P_m^o = 1$.

Seriem significo termino generali, cui Σ proponitur, ubi quantitates a terminorum numero dependentes litteris germanis denotantur, pro quibus zero et omnes numeri integri poni possunt; ergo exempli gratia

$$(a+b)^m = \Sigma P_m^{\mathfrak{a}} a^{m-\mathfrak{a}} b^{\mathfrak{a}} =$$
$$a^m + m a^{m-1} b + \frac{m.m-1}{1.\ 2} a^{m-2} b^2 + \ldots \frac{m.\ m-1\ \ldots\ m-\mathfrak{a}+1}{1.\ 2\ \ldots\ldots\ \mathfrak{a}} a^{m-\mathfrak{a}} b^{\mathfrak{a}} \ldots$$

Si series eo modo mutatur, ut omnes termini aeque afficiantur, hoc tantum in generali termino significandum est, et termini generalis factor a termini numero non dependens ante summae signum poni potest; sicuti

$$\text{Cos. } m\,\varphi \sum P_m^a \text{ Cos. } a\,\varphi = \sum \text{Cos. } m\,\varphi\, P_m^a \text{ Cos. } a\,\varphi.$$

Praeterea littera i semper utemur ad $\sqrt{-1}$ et $\overline{\Pi\, n}$ ad productum omnium numerorum integrorum a 1 usque ad n exprimendum.

§. 3.

Euler in subsidio calculi sinuum, quae dissertatio tomo quinto novorum commentariorum academiae Petropolitanae (1755 pag. 164) inserta est, hac fere methodo utitur. Ponit nempe $y = \text{Cos. } \varphi + i \text{ Sin. } \varphi$ et $z = \text{Cos. } \varphi - i \text{ Sin. } \varphi$; ergo $y + z = 2 \text{ Cos. } \varphi$ et $y\, z = 1$. Itaque

$$(2 \text{ Cos. } \varphi)^m = (y + z)^m$$

quod secundum notam binomii formulam explicatum, fit

$$(2 \text{ Cos. } \varphi)^m = \sum P_m^a\, y^{m-a}\, z^a = \sum P_m^a\, y^{m-2a}$$

quia $z = \frac{1}{y} = y^{-1}$ et $z^a = y^{-a}$. Pro y valore restituto transit illa formula in

$$(2 \text{ Cos. } \varphi)^m = \sum P_m^a\, (\text{Cos. } \varphi + i \text{ Sin. } \varphi)^{m-2a}$$

Secundum theorema a Moivre propositum est

$(\text{Cos. } x + i \text{ Sin. } x)^n = \text{Cos. } nx + i \text{ Sin. } nx$, ergo in nostro casu

$$(\text{Cos. } \varphi + i \text{ Sin. } \varphi)^{m-2a} = \text{Cos. } (m - 2a)\,\varphi + i \text{ Sin. } (m - 2a)\,\varphi$$

$$\text{et } (2 \text{ Cos. } \varphi)^m = \sum P_m^a \left\{ \text{Cos. } (m - 2a)\,\varphi + i \text{ Sin. } (m - 2a)\,\varphi \right\}$$

$$= \sum P_m^a \text{ Cos. } (m - 2a)\,\varphi + i \sum P_m^a \text{ Sin. } (m - 2a)\,\varphi \qquad (1)$$

Si vero $(y + z)^m$ secundum ascendentes potestates y explicatur, simili ratione invenitur

$$(2 \text{ Cos. } \varphi)^m = (z + y)^m = \sum P_m^a\, z^{m-a}\, y = \sum P_m^a\, z^{m-2a}$$

$$= \sum P_m^a \left\{ \text{Cos. } \varphi - i \text{ Sin. } \varphi \right\}^{m-2a}$$

ergo $(2 \text{ Cos. } \varphi)^m = \sum P_m^a \text{ Cos. } (m - 2a)\,\varphi - i \sum P_m^a \text{ Sin. } (m - 2a)\,\varphi$ (2)

Summas ita inventas (1 et 2) Euler additione conjungit, unde

$$2\, (2 \text{ Cos. } \varphi)^m = 2 \sum P_m^a \text{ Cos. } (m - 2a)\,\varphi$$

quia sinuum series propter contrarium signum invicem se tollunt. Ex quibus sequitur

$$(2\ \mathrm{Cos.}\ \varphi)^m = \Sigma\, P_m^a\, \mathrm{Cos.}\,(m - 2a)\,\varphi$$
$$= \mathrm{Cos.}\, m\varphi + m\ \mathrm{Cos.}\,(m - 2)\,\varphi + \frac{m.\ m-1}{1.\ \ 2}\,\mathrm{Cos.}\,(m - 4)\,\varphi \ldots \qquad (3)$$

Secundum quam solutionem inventam Euler exempla nonnulla explicat, contentus, quod cum notis formulis consentiant series, si pro m parvos numeros integros ponit.

§. 4.

Lagrange in libro suo: „Leçons sur le calcul des fonctions“ undecima lectione calculi differentialis et indefinitorum coefficientium auxilio usus, sequenti methodo invenit aequationem (3. §. 3.). — Ponit nempe

$y = (\mathrm{Cos.}\ \varphi)^m$, ex qua aequatione differentiata procedit

$$\frac{dy}{d\varphi} = -m\ \mathrm{Cos.}\ \varphi^{m-1}\ \mathrm{Sin.}\ \varphi$$

et utraque parte per Cos. φ multiplicata

$$\frac{dy}{d\varphi}\ \mathrm{Cos.}\ \varphi = -m\ \mathrm{Cos.}\ \varphi^m\ \mathrm{Sin.}\ \varphi$$

aut $0 = \frac{dy}{d\varphi}\ \mathrm{Cos.}\ \varphi + m\ y\ \mathrm{Sin.}\ \varphi$. — Res jam in eo est, ut series inveniatur, quae aequationibus et $y = \mathrm{Cos.}\ \varphi^m$ et $o = \frac{dy}{d\varphi}\,\mathrm{Cos.}\ \varphi + m\ y\ \mathrm{Sin.}\ \varphi$ (4) aequo modo respondeat. Ad quem finem Lagrange fingit seriem

$y = A\ \mathrm{Cos.}\ n\ \varphi + B\ \mathrm{Cos.}\,(n-1)\,\varphi + C\ \mathrm{Cos.}\,(n-2)\,\varphi + D\ \mathrm{Cos.}\,(n-3)\,\varphi + E\ \mathrm{Cos.}\,(n-4)\,\varphi \ldots\ldots$ ex qua procedit

$$-\frac{dy}{d\varphi} = n\ A\ \mathrm{Sin.}\ n\varphi + (n-1)\ B\ \mathrm{Sin.}\,(n-1)\,\varphi$$
$$+ (n-2)\ C\ \mathrm{Sin.}\,(n-2)\,\varphi + (n-3)\ D\ \mathrm{Sin.}\,(n-3)\,\varphi$$
$$+ E\ (n-4)\ \mathrm{Sin.}\,(n-4)\,\varphi \ldots\ldots$$

Quibus valoribus in locum y et $\frac{dy}{d\varphi}$ positis aequatio (4) transit in

$$o = -\mathrm{Cos.}\ \varphi \left\{ n\ A\ \mathrm{Sin.}\ n\ \varphi + (n-1)\ B\ \mathrm{Sin.}\,(n-1)\,\varphi \right.$$
$$+ (n-2)\ C\ \mathrm{Sin.}\,(n-2)\,\varphi + (n-3)\ D\ \mathrm{Sin.}\,(n-3)\,\varphi$$
$$\left. + (n-4)\ E\ \mathrm{Sin.}\,(n-4)\,\varphi \ldots \right\} + m\ \mathrm{Sin.}\ \varphi \left\{ A\ \mathrm{Cos.}\ n\ \varphi \right.$$
$$+ B\ \mathrm{Cos.}\,(n-1)\,\varphi + C\ \mathrm{Cos.}\,(n-2)\,\varphi + D\ \mathrm{Cos.}\,(n-3)\,\varphi$$
$$\left. + E\ \mathrm{Cos.}\,(n-4)\,\varphi \ldots \right\}$$

Ut vero series oriatur, quae secundum unam variabilis φ functionem procedat, termini omnes dissolvendi sunt. Nam

$$\text{Cos.}\, \alpha\varphi\, \text{Sin.}\, \varphi = \tfrac{1}{2}\, \text{Sin.}\, (\alpha + 1)\, \varphi - \tfrac{1}{2}\, \text{Sin.}\, (\alpha - 1)\, \varphi$$
$$\text{Sin.}\, \alpha\varphi\, \text{Cos.}\, \varphi = \tfrac{1}{2}\, \text{Sin.}\, (\alpha + 1)\, \varphi + \tfrac{1}{2}\, \text{Sin.}\, (\alpha - 1)\, \varphi$$

Omnibus illius seriei terminis secundum has formulas mutatis et $\frac{1}{2}$ omisso, quia in omnibus terminis occurrit, fit

$$\begin{aligned}
o = & - n\, A\, \text{Sin.}\, (n+1)\, \varphi - (n-1)\, B\, \text{Sin.}\, n\varphi - n\, A\, \text{Sin.}\, (n-1)\, \varphi \\
& \qquad - (n-2)\, C\, \text{Sin.}\, (n-1)\, \varphi \\
& + m\, A\, \text{Sin.}\, (n+1)\, \varphi \quad + m\, B\, \text{Sin.}\, n\varphi - m\, A\, \text{Sin.}\, (n-1)\, \varphi \\
& \qquad + m\, C\, \text{Sin.}\, (n-1)\, \varphi \\
& - (n-1)\, B\, \text{Sin.}\, (n-2)\, \varphi \\
& - (n-3)\, D\, \text{Sin.}\, (n-2)\, \varphi - (n-3)\, C\, \text{Sin.}\, (n-3)\, \varphi - (n-3)\, D\, \text{Sin.}\, (n-4)\, \varphi \\
& \qquad - (n-4)\, E\, \text{Sin.}\, (n-3)\, \varphi - (n-5)\, F\, \text{Sin.}\, (n-4)\, \varphi \\
& - m\, B\, \text{Sin.}\, (n-2)\, \varphi \ldots \\
& + m\, D\, \text{Sin.}\, (n-2)\, \varphi - m\, C\, \text{Sin.}\, (n-3)\, \varphi - m\, D\, \text{Sin.}\, (n-4)\, \varphi \\
& \qquad + m\, E\, \text{Sin.}\, (n-3)\, \varphi + m\, F\, \text{Sin.}\, (n-4)\, \varphi
\end{aligned}$$

Si termini eodem sinu multiplicati pro se evanescere debent, coefficientes auxilio primi A definiri possunt. Nam inde hae aequationes oriuntur: $o = A\,(m-n)$; $o = B\,(m-n+1)$; $o = C\,(m-n+2) - A\,(m+n)$; $o = D\,(m-n+3) - B\,(m+n-1)$; $o = E\,(m-n+4) - C\,(m+n-2)$; $o = F\,(m-n+5) - D\,(m+n-3)$.

Primae aequationi satisfacit $m = n$; porro $B = o$, $D = o$, $F = o$, eodemque modo evanescunt omnes coefficientes imparium terminorum. Fit vero $C = A\frac{m+n}{m-n+2} = A\,m$, quia $m = n$.

$$E = C\frac{m+n-2}{m-n+4} = \frac{2(m-1)}{4}\, C = \frac{m-1}{2}\, C = \frac{m.\, m-1}{1.\ 2}\, A.$$

Ita coefficientes binomiales invenimus etiam in sequentibus terminis, si calculum longius producimus. Pervenimus ergo ad

$$y = \text{Cos.}\, \varphi^m = A \sum P_m^a\, \text{Cos.}\, (m-2a)\, \varphi.$$

Ut vero A constituatur, ponit Lagrange $\varphi = o$, quo casu $\text{Cos.}\, \varphi = 1$, ergo etiam

$$y = \text{Cos.}\, \varphi^m = 1 \text{ et } \sum P_m^a\, \text{Cos.}\, (m-2a)\, \varphi = \sum P_m^a\, \text{Cos.}\, o = \sum P_m^a = 2^m;$$

ex quo concludit $1 = A\ 2^m$ et $A = \frac{1}{2^m}$; itaque

$(\text{Cos.}\ \varphi)^m = \frac{1}{2^m} \Sigma P_m^\alpha\ \text{Cos.}\ (m - 2\ \alpha)\ \varphi$, et quod idem est

$(2\ \text{Cos.}\ \varphi)^m = \Sigma P_m^\alpha\ \text{Cos.}\ (m - 2\ \alpha)\ \varphi$, sicuti Euler invenit.

§. 5.

Poisson (Correspondence sur l'école polytechnique par Hachette Tom. II p. 212) primus ostendit, Euleri formulam universe valere non posse, quia $(2\ \text{Cos.}\ \varphi)^m$, si m numerus fractus sit, varios valores habeat, quorum (1 et 2 § 3) modo duo sint, in unum autem corruentes, m numero integro. Nam tunc $(2\ \text{Cos.}\ \varphi)^m$ unum tantummodo valorem habet; sed ut illi duo simul recti sint, necesse

$$\Sigma P_m^\alpha\ \text{Sin.}\ (m - 2\alpha)\ \varphi = 0 \text{ esse debet.}$$

Quam seriem nihilo aequalem esse, ita demonstrari potest. — Series hoc casu est finita et terminus ultimus $= P_m^m \text{Sin.}\ (m - 2\ m)\ \varphi = \text{Sin.}\ (-m\ \varphi) = -\ \text{Sin.}\ m\ \varphi$; ergo primo aequalis, sed contrario signo affectus; itaque summando invicem se tollunt. Eandem rationem habent termini aeque ab initio et fine distantes. Nam terminus γ ab initio $= P_m^\gamma\ \text{Sin.}\ (m - 2\ \gamma)\ \varphi$ et a fine $= P_m^{m-\gamma}\ \text{Sin.}\ (m - 2\ (m - \gamma))\ \varphi$

$$= P_m^\gamma\ \text{Sin.}\ (-m + 2\ \gamma)\ \varphi = -P_m^\gamma\ \text{Sin.}\ (m - 2\ \gamma)\ \varphi$$

et summa amborum terminorum $= 0$. Si vero summando terminus medius restat, quod fit m numero pari, quia series $m + 1$ terminos complectitur, ille est $= P_m^{\frac{1}{2}m}\ \text{Sin.}\ (m - 2\frac{m}{2})\ \varphi = P_m^{\frac{1}{2}m}\ \text{Sin.}\ 0\ \varphi = 0$.

Ex quibus elucet, $\Sigma P_m^\alpha\ \text{Sin.}\ (m - 2\alpha)\ \varphi = 0$ et $\Sigma P_m^\alpha\ \text{Cos.}\ (m - 2\alpha)\ \varphi = (2\ \text{Cos.}\ \varphi)^m$ si m numerus integer est. — Exemplum, quo Poisson illius formulae errorem detexit, est $\varphi = \pi$ et $m = \frac{1}{3}$, ergo

$$(2\ \text{Cos.}\ \pi)^{\frac{1}{3}} = (-2)^{\frac{1}{3}} = \Sigma P_{\frac{1}{3}}^\alpha\ \text{Cos.}\ (\tfrac{1}{3} - 2\ \alpha)\ \pi = \text{Cos.}\ \tfrac{1}{3}\ \pi\ \Sigma P_{\frac{1}{3}}^\alpha$$

Est vero $\text{Cos.}\ \frac{1}{3}\pi = \frac{1}{2}$, ergo secundum formulam esse deberet $(-2)^{\frac{1}{3}} = \frac{1}{2}.2^{\frac{1}{3}}$, quod nullo modo esse potest.

§. 6.

Etiam alio modo ostendi potest $(2\ \text{Cos.}\ \varphi)^m = \Sigma P_m^\alpha\ \text{Cos.}\ (m - 2\ \alpha)\ \varphi$

si m numerus integer est. Nam formulam rectam esse pro $m = 1$, $= 2$, $= 3$, facile intelligitur; exempli gratia $(2 \text{ Cos. } \varphi)^3 =$

$$= \text{Cos. } 3\varphi + 3 \text{ Cos. } \varphi + 3 \text{ Cos. } (-\varphi) + \text{Cos. } -3\varphi$$
$$= 2 (\text{Cos. } 3\varphi + 3 \text{ Cos. } \varphi)$$

Sed $\text{Cos. } 3\varphi = \text{Cos. } \varphi^3 - 3 \text{ Cos. } \varphi \text{ Sin. } \varphi^2$, ergo $\text{Cos. } 3\varphi + 3 \text{ Cos. } \varphi = \text{Cos. } \varphi^3 + 3 \text{ Cos. } \varphi (1 - \text{Sin. } \varphi^2) = \text{Cos. } \varphi^3 + 3 \text{ Cos. } \varphi \text{ Cos. } \varphi^2 = 4 \text{ Cos. } \varphi^3$, ergo $2 (\text{Cos. } 3\varphi + 3 \text{ Cos. } \varphi) = 8 \text{ Cos. } \varphi^3 = (2 \text{ Cos. } \varphi)^3$. Demonstrare experiemur, formulam illam ad $m + 1$ pertinere, si pro m recta habetur. Nempe

$$(2 \text{ Cos. } \varphi)^m = \Sigma P_m^a \text{ Cos. } (m - 2a)\varphi; \text{ ergo}$$

$$(2 \text{ Cos. } \varphi)^{m+1} = 2 \text{ Cos. } \varphi \, \Sigma P_m^a \text{ Cos. } (m - 2a)\varphi,$$

aut $(2 \text{ Cos. } \varphi)^{m+1} = 2 \text{ Cos. } \varphi \text{ Cos. } m\varphi + 2m \text{ Cos. } \varphi \text{ Cos. } (m-2)\varphi + 2 \text{ Cos. } \varphi \frac{m.m-1}{1.\ 2} \text{ Cos. } (m-4)\varphi \ldots . \; 2 \text{ Cos. } \varphi \, P_m^{'a} \text{ Cos. } (m - 2a)\varphi + 2 \text{ Cos. } \varphi \, P_m^{a+1} \text{ Cos. } \{m - 2(a+1)\} \varphi$.

Est vero $2 \text{ Cos. } \varphi \text{ Cos. } \alpha\varphi = \text{Cos. } (\alpha + 1)\varphi + \text{Cos. } (\alpha - 1)\varphi$, secundum quam formulam omnes termini dissolvi possunt.

Ergo $2 \text{ Cos. } \varphi \text{ Cos. } m\varphi = \text{Cos. } (m+1)\varphi + \text{Cos. } (m-1)\varphi$

$$2m \text{ Cos. } \varphi \text{ Cos. } (m-2)\varphi = m \text{ Cos. } (m-1)\varphi + m \text{ Cos. } (m-3)\varphi$$

$$2 \frac{m.m-1}{1.\ 2.} \text{ Cos. } \varphi \text{ Cos. } (m-4)\varphi = + \frac{m.m-1}{1.\ 2.} \text{Cos. } (m-3)\varphi + \frac{m.m-1}{1.\ 2} \text{Cos. } (m-5)\varphi$$

$$2 \frac{m.m-1.m-2}{1.\ 2.\ 3} \text{ Cos. } \varphi \text{ Cos. } (m-6)\varphi = + \frac{m.m-1.m-2}{1.\ 2.\ 3.} \text{ Cos. } (m-5)\varphi + \frac{m.m-1.m-2}{1.\ 2.\ 3} \text{ Cos. } (m-7)\varphi$$

. . .

$$2 P_m^a \text{ Cos. } (m-2a)\varphi \text{ Cos. } \varphi = P_m^a \text{ Cos. } (m-2a+1)\varphi + P_m^a \text{ Cos. } (m-2a-1)\varphi$$

$$2 P_m^{a+1} \text{ Cos. } (m-2a-2)\varphi \text{ Cos. } \varphi = + P_m^{a+1} \text{ Cos. } (m-2a-1)\varphi + P_m^{a+1} \text{ Cos. } (m-2a-3)\varphi \ldots .$$

. . .

Si termini eodem cosinu affecti summantur, primus fit Cos. $(m+1)\varphi$, secundus $(m+1)$ Cos. $(m-1)\varphi = (m+1)$ Cos. $(m+1-2)\varphi$, tertius $\left(m+\frac{m.\, m-1}{1.\ 2}\right)$ Cos. $(m-3)\varphi = \frac{m+1.m}{1.\ 2}$ Cos. $(m+1-4)\varphi$, quartus $\left\{\frac{m.\, m-1}{1.\ 2}+\frac{m.m-1.\, m-2}{1.\ 2.\ 3}\right\}$ Cos. $(m-5)\varphi = \frac{m+1.m.\, m-1}{1.\ 2.\ 3}$ Cos. $(m+1\text{-}6)\varphi$

Itaque videmus, primos quidem terminos eandem regulam sequi, quae etiam apud terminos generales locum habet. Nam notum est
$P^{a}_{m} + P^{a+1}_{m} = P^{a+1}_{m+1}$;
ergo $(P^{a}_{m} + P^{a+1}_{m})$ Cos. $(m-2a-1)\varphi = P^{a+1}_{m+1}$ Cos. $\{m+1\text{-}2(a+1)\}\varphi$, qui terminus omnino regulae respondet. Facile perspici potest, simili modo P^{a+1}_{m} Cos. $(m-2a-3)\varphi$ cum alio termino conjunctum efficere
P^{a+2}_{m+1} Cos. $\{m+1-2(a+2)\}\varphi$.

Ex quibus elucet,

$$(2\ \text{Cos.}\ \varphi)^{m+1} = P^{a}_{m+1}\ \text{Cos.}\ (m+1-2a)\varphi.$$

Quum vero formula pro $m = 3$ locum habeat, etiam pro $m = 4$, $= 5$ et pro quocunque numero integro convenientis rationis esse apparet.

§. 7.

Quod autem ad fractos exponentes attinet, in antecessum exemplo probaturus sum, Euleri formulam nonnunquam rectam summam exhibere. Ponamus $\varphi = 63°$ et $m = \frac{7}{9}$. Ergo

$$(2\ \text{Cos.}\ 63°)^{\frac{7}{9}} = \text{Cos.}\ 49° + \frac{7}{9}\text{Cos.}\ 77° - \frac{7}{81}\text{Cos.}\ (\pi+23) + \frac{77}{3.9^3}\text{Cos.}(2\pi\text{-}31)$$
$$- \frac{5.77}{3.\ 9^4}\text{Cos.}\ (3\pi-85) + \frac{77.29}{3.\ 9^5}\text{Cos.}(3\pi+41) - \frac{77.19.29}{9^7}\text{Cos.}\ (4\pi\ \text{-}13)$$
$$+ \frac{11.19.29.47}{9^8}\text{Cos.}(5\pi-67) - \frac{77.19.29.47}{9^9}\text{Cos.}\ (5\pi+59)$$
$$+ \frac{77.19.29.47.65.}{9^{11}}\text{Cos.}\ (6\pi+5) - \frac{77.19.29.47.13.37}{9^{12}}\text{Cos.}\ (7\pi-49)$$
$$+ \frac{7.13.83.\acute{a}}{9^{13}}\text{Cos.}\ (7\pi+77) - \frac{7.13.23.83.\acute{a}}{3.9^{14}}\text{Cos.}\ (8\pi+23)$$

$$+\frac{7.23.83.101.\acute{\alpha}}{3.9^{15}}\,\mathrm{Cos.}(9\pi-31)-\frac{23.\acute{\alpha}.83.101.55}{3.9^{16}}\,\mathrm{Cos.}(10\pi-85)$$

$$+\frac{11.\acute{\alpha}.\acute{\beta}.119}{9^{18}}\,\mathrm{Cos.}(10\pi+41)-\frac{\acute{\alpha}.\acute{\beta}.11.119.8}{9^{19}}\,\mathrm{Cos.}(11\pi-13)$$

$$+\frac{\acute{\alpha}.\acute{\beta}.8.11.7.137}{9^{20}}\,\mathrm{Cos.}(12\pi-67)-\frac{\acute{\alpha}.\acute{\beta}.4.77.137.146}{9^{22}}\,\mathrm{Cos.}(12\pi+59)$$

$$+\frac{4.77.29.37.47.\acute{\beta}.137.146.155}{9^{23}}\,\mathrm{Cos.}(13\pi+5)$$

ubi brevitatatis causa $\acute{\alpha} = 958189 = 19.29.37.47$ et $\acute{\beta}$ pro $192809 = 23.83.101$ pono. — Arcus quidem primo excepto omnes sunt negativi; quia vero eorum cosinus eundem valorem habent, negationis signum omissum est. — Sinibus autem contrarium signum datur propter negativos arcus, ubi modo numerum coefficientium significabo.

Terminorum illorum ratione conficta efficitur

0	= 0,6560590	—	5	= 0,0095134	
1	= 0,1753651		6	= 0,0086431	
2	= 0,0795498		7	= 0,0025857	
3	= 0,0301792		11	= 0,0006405	
4	= 0,0017048		12	= 0,0022326	
8	= 0,0026509		13	= 0,0017947	
9	= 0,0041147		14	= 0,0001593	
10	= 0,0022280		18	= 0,0010463	
15	= 0,0012160				
16	= 0,0013955				
17	= 0,0005001				
	= 0,9549631	—		0,0272835	= 0,9276792

Si vero alio modo potestatis valorem computamus, impetramus $(2\ \mathrm{Cos.}\ 63^\circ)^{\frac{7}{9}} = 0,9276691$; ubi primi numeri consentiunt, ceteri autem eo magis consentient, quo longius seriem producimus; sed ad priores numeros termini sequentes non pertinent.

Eodem arcu utemur pro sinuum serie, $o = \Sigma P_{\frac{7}{9}}^{a}$ Sin. $(\frac{7}{9} - 2a)\, 63^\circ$.

	Sin. 49°	= 0,7547096	—	P [1]	Sin. 77	= 0,7578434
P [3]	Sin. 31	= 0,0181335		-- [2]	Sin. 23	= 0,0337668
-- [4]	Sin. 85	= 0,0194856		-- [6]	Sin. 13	= 0,0019954
-- [5]	Sin. 41	= 0,0082698		-- [7]	Sin. 67	= 0,0060915
-- [10]	Sin. 49	= 0,0025630		-- [8]	Sin. 59	= 0,0044119
-- [11]	Sin. 77	= 0,0027742		-- [9]	Sin. 5	= 0,0003600
-- [12]	Sin. 23	= 0,0009477		-- [13]	Sin. 13	= 0,0010784
-- [17]	Sin. 67	= 0,0011805		-- [14]	Sin. 85	= 0,0018209
				-- [16]	Sin. 41	= 0,0010571

= 0,8083861 — 0,8084254. — E convenientibus numeris apparet, seriem valori = o appropinquare.

§. 8.

Nonnulli autem probare conati sunt, seriei summam ΣP_m^a Sin. $(m - 2a)\,\varphi$ in universum = 0 fieri.

Plana, Astromiae professor Turinas (Annales de mathematiques par Gergonne Tom. VI 1820) contendit, seriem illam = 0 esse, unico casu $\varphi = \pi$ excepto. Ad hoc demonstrandum ponit Sin. $\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$ ergo

$$\Sigma P_m^a \text{ Sin. } (m - 2a)\,\varphi = \Sigma P_m^a \frac{e^{i(m-2a)\varphi} - e^{-i(m-2a)\varphi}}{2i}$$

$$= \Sigma P_m^a \frac{e^{m\varphi i} \cdot e^{-2a\varphi i}}{2i} - \Sigma P_m^a \frac{e^{-m\varphi i} \cdot e^{2a\varphi i}}{2i}$$

ergo primae seriei termini omnes factorem $\frac{e^{im\varphi}}{2i}$, secundae factorem $\frac{e^{-m\varphi i}}{2i}$ complectuntur; quamobrem

$$\Sigma P_m^a \text{ Sin. } (m - 2a)\,\varphi = \frac{e^{m\varphi i}}{2i} \Sigma P_m^a e^{-2a\varphi i} - \frac{e^{-m\varphi i}}{2i} \Sigma P_m^a e^{2a\varphi i}$$

$$= \frac{e^{im\varphi}}{2i}(1 + e^{-2\varphi i})^m - \frac{e^{-m\varphi i}}{2i}(1 + e^{2\varphi i})^m$$

Quum vero $e^{\pm 2\varphi i} = \text{Cos.}\,2\varphi \pm i\,\text{Sin.}\,2\varphi = 2\,\text{Cos.}^2\varphi - 1 + 2i\,\text{Sin.}\,\varphi\,\text{Cos.}\,\varphi$

$$= 2\,\text{Cos.}\,\varphi\,\left\{\text{Cos.}\,\varphi + i\,\text{Sin.}\,\varphi\right\} - 1$$

formula illa transit in

$$\Sigma\, P_m^a\,\text{Sin.}\,(m-2a)\varphi = \frac{e^{m\varphi i}}{2i}(\text{Cos.}\,\varphi - i\,\text{Sin.}\,\varphi)^m\,(2\,\text{Cos.}\,\varphi)^m$$

$$- \frac{e^{-m\varphi i}}{2i}(\text{Cos.}\,\varphi + i\,\text{Sin.}\,\varphi)^m\,(2\,\text{Cos.}\,\varphi)^m$$

$$= \frac{(2\,\text{Cos.}\,\varphi)^m}{2i}(\text{Cos.}\,m\varphi + i\,\text{Sin.}\,m\varphi)(\text{Cos.}\,m\varphi - i\,\text{Sin.}\,m\varphi)$$

$$- \frac{(2\,\text{Cos.}\,\varphi)^m}{2i}(\text{Cos.}\,m\varphi - i\,\text{Sin.}\,m\varphi)(\text{Cos.}\,m\varphi + i\,\text{Sin.}\,m\varphi) = 0$$

Quae demonstratio bene se habere videtur, sed exceptio $\varphi = \pi$ ex ea non elucet. (cf. §§ 23, 24).

§. 9.

Deflers hac methodo utitur. (Lacroix traité du calcul différentiel et integral Tom. III. 621). Series cosinum exprimens pares tantum arcus potestates complectitur; igitur potestas quaedam ita explicata solas pares potestates arcus continere potest. Ex quo Deflers concludit, in $(2\,\text{Cos.}\,\varphi)^m = \Sigma\, P_m^a\,\text{Cos.}\,(m-2a)\,\varphi + i\,\Sigma\,\text{Sin.}\,(m-2a)\,\varphi$ secundam partem evanescere debere, quia sinus simili modo explicati impares potestates exhibeant.

Praeterea Deflers alia ratione idem probare conatur. Nempe si omnes termini seriei $\Sigma\, P_m^a\,\text{Sin.}\,(m-2a)\,\varphi$ secundum potestates arcuum explicantur, fit

$$= \varphi\,(m + P_m^1\,(m-2) + P_m^2\,(m-4)\ldots\ldots + P_m^a\,(m-2a)\ldots\ldots)$$

$$- \frac{\varphi^3}{6}\left\{m^3 + P_m^1\,(m-2)^3 + P_m^3\,(m-4)^3\ \ldots\ldots + P_m^a\,(m-2a)^3\ \ldots\right\}$$

$$\vdots$$

$$\frac{(-1)^b \varphi^{2b+1}}{\Pi\, 2b+1} \left\{ m^{2b+1} + P_m^1 (m-2)^{2b+1} \ldots\ldots + P_m^a (m-2a)^{2b+1} \ldots\ldots \right\} \text{etc.}$$

$$= \Sigma \frac{(-1)^b \varphi^{2b+1}}{\Pi\, 2b+1} \Sigma P_m^a (m-2a)^{2b+1}$$

Ad quam summam inveniendam inquirit functionem

$T_p = \Sigma P_m^a (m-2a)^p t^{m-2a}$

$$\frac{dT_p}{dt} = \Sigma P_m^a (m-2a)^{p+1} \frac{t^{m-2a}}{t} = \frac{T_{p+1}}{t}. \text{ Ergo}$$

$\frac{dT_0}{dt} = \frac{T_1}{t}$; $\frac{dT_1}{dt} t = T_2$, ubi $T_0 = \Sigma P_m^a t^{m-2a} = (t + t^{-1})^m$

et si $t + t^{-1} = z$ ponimus, $T_0 = z^m$; porro $\frac{dz}{dt} = 1 - t^{-2}$

ubi $v = \frac{dz}{dt} t = t - t^{-1}$ assumitur; itaque $T_0 = z^m$; $T_1 = m z^{m-1} v$;

$$T_2 = m\, m-1.\, z^{m-2} v^2 + m z^m,\ T_3 = m.\, m-1.\, m-2.\, z^{m-3} v^3$$
$$+ (2 m.\, m-1 + m^2)\, z^{m-1} v,$$

quod longius produci potest. Elucet autem, seriem ita explicatam, p numero impari, e terminis compositam esse, qui quamlibet potestatem v continent. Sed $t = 1$ posito, hae series in illas transeunt, in quas supra $\Sigma P_m^a \text{Sin.} (m-2a)\varphi$ evoluta est. Quum autem $t = 1$ posito, $v = t - t^{-1} = o$ fiat, omnes termini aliquam potestatem v complexi evanescere debent, ergo etiam series illae et $\Sigma P_m^a \text{Sin.} (m-2a)\varphi = o$.

§. 10.

Lacroix (Traité III p. 616.) methodum, qua Lagrange usus est, ad formulam

$(2 \text{Cos.} \varphi)^m = \Sigma P_m^a \text{Cos.} (m-2a\varphi + i \Sigma P_m^a \text{Sin.} (m-2a)\varphi = X + i Y = 2^m$ adhibet. Differentiando efficitur

$$o = m X \text{Sin.} \varphi + \frac{dX}{d\varphi} \text{Cos.} \varphi + i \left\{ n Y \text{Sin.} \varphi + \frac{dY}{d\varphi} \text{Cos.} \varphi \right\}$$

ubi utraque pars, et realis et imaginaria, pro se evanescere debet; ergo $o = m X \text{Sin.} \varphi + \frac{dX}{d\varphi} \text{Cos.} \varphi$ et $o = m Y \text{Sin.} \varphi + \frac{dY}{d\varphi} \text{Cos.} \varphi$.

Primam aequationem explet $y = A \Sigma P_m^a \text{Cos.} (m - 2a) \varphi$, ut Lagrange probavit. Simili ratione secundae satisfacit $y = BY = B\Sigma P_m^a \text{Sin.}(m-2a)\varphi$; ergo ex una aequatione differentiali primi ordinis duae constantes quantitates arbitrariae prodire videntur, et quidem contra leges calculi differentialis. Itaque inter se cohaereant, necesse est. Nam

$$\frac{dX}{d\varphi} + m X \text{tang.} \varphi = o \text{ et } \frac{dY}{d\varphi} + m Y \text{tang.} \varphi = o$$

ergo $m \text{tang.} \varphi . d\varphi = \frac{dX}{X} = \frac{dY}{Y}$; porro $\log. X = \log Y + \log. C$

itaque $\frac{X}{Y} = C$; et quod inde sequitur

$$y = A X + B Y = (C A + B) Y \text{ aut } = (\frac{A}{C} + B) X.$$

quae serierum conjunctio postea clarius elucebit. — Sed $\varphi = o$ assumpto, fit $A = \frac{1}{2^m}$ et $B = \frac{1}{0}$, id est infinitum. Ex quo Lacroix concludit secundum valorem Y aequationi quidem, multo longius patenti, sed non speciali casui satisfacere; quamob causam ponendum esse

$$(2 \text{Cos.} \varphi)^m = \Sigma P_m^a \text{Cos.} (m - 2a) \varphi.$$

§. 11.

Poisson in detegendo errore non acquievit, sed studuit, ut formulam, omnes potestatis valores exprimentem et pro quocunque arcu valentem inveniret. (Ferussac Bulletin des sciences mathématiques IV 1825. p. 140) Euleri calculum (§ 3) adhibet usque ad

$$(2 \text{Cos.} \varphi)^m = \Sigma P_m^a (\text{Cos.} \varphi + i \text{Sin.} \varphi)^m$$

Sed $(\text{Cos.} \varphi + i \text{Sin.} \varphi)^m$ non $= \text{Cos.} m\varphi + i \text{Sin.} m \varphi$ sumit, sed ut totidem varios valores exhibeat $= \text{Cos.} (\varphi + 2 \alpha \pi) m + i \text{Sin.} (\varphi + 2 \alpha \pi) m$, ubi si m numerus fractus $= \frac{r}{s}$, pro α omnes numeri integri ponendi sunt ab o usque ad $s - 1$; qua ratione dextra et sinistra manu s varios valores exprimuntur. — Quo theoremate ad nostrum casum adhibito, fit

$$(2 \operatorname{Cos.} \varphi)^m = \Sigma P_m^a \left\{ \operatorname{Cos.} (\varphi + 2 \alpha \pi)(m - 2 a) + i \operatorname{Sin.} (\varphi + 2 \alpha \pi)(m - 2 a) \right\}$$
$$= \Sigma P_m^a \left\{ \operatorname{Cos.} (m - 2 a) \varphi \operatorname{Cos.} 2 m \alpha \pi - \operatorname{Sin.} (m - 2 a) \varphi \operatorname{Sin.} 2 m \alpha \pi \right\}$$
$$+ i \Sigma P_m^a \left\{ \operatorname{Sin.} (m - 2 a) \varphi \operatorname{Cos.} 2 m \alpha \pi + \operatorname{Cos.} (m - 2 a) \varphi \operatorname{Sin.} 2 m \alpha \pi \right\}$$
$$= \Sigma P_m^a \left\{ \operatorname{Cos.} (m - 2 a) \varphi + i \operatorname{Sin.} (m - 2 a) \varphi \right\}$$
$$\left\{ \operatorname{Cos.} 2 m \alpha \pi + i \operatorname{Sin.} 2 m \alpha \pi \right\} \qquad (5)$$

Et ipso calculo quidem prodit Cos. $(m - 2 a)\, 2 \alpha \pi$ et Sin. $(m - 2 a)\, 2 \alpha \pi$, sed secundum nota trigonometrica theoremata $=$ Cos. $2 m \alpha \pi$ et Sin. $2 m \alpha \pi$. Si vero Q realem potestatis valorem significat, etiam esse debet

$$(2 \operatorname{Cos.} \varphi)^m = Q \left\{ \operatorname{Cos.} 2 m \acute{\alpha} \pi + i \operatorname{Sin.} 2 m \acute{\alpha} \pi \right\}. \qquad (6)$$

In aequationibus (6 et 5) dextra manu et reales et imaginarii termini sibi aequales esse debent; ergo

$$Q \operatorname{Cos.} 2 m \acute{\alpha} \pi = \operatorname{Cos.} 2 m \alpha \pi \, \Sigma P_m^a \operatorname{Cos.} (m - 2 a) \varphi$$
$$- \operatorname{Sin.} 2 m \alpha \pi \, \Sigma P_m^a \operatorname{Sin.} (m - 2 a) \varphi$$
$$Q \operatorname{Sin.} 2 m \acute{\alpha} \pi = \operatorname{Sin.} 2 m \alpha \pi \, \Sigma P_m^a \operatorname{Cos.} (m - 2 a) \varphi$$
$$+ \operatorname{Cos.} 2 m \alpha \pi \, \Sigma P_m^a \operatorname{Sin.} (m - 2 a) \varphi$$

Si aequationes, primam per Cos. $2 m \alpha \pi$ et secundam per Sin. $2 m \alpha \pi$ multiplicatas, additione conjungo, accipio

$$Q \left\{ \operatorname{Cos.} 2 m \alpha \pi \operatorname{Cos.} 2 m \acute{\alpha} \pi + \operatorname{Sin.} 2 m \alpha \pi \operatorname{Sin.} 2 m \acute{\alpha} \pi \right\} =$$
$$= (\operatorname{Cos.} 2 m \alpha \pi^2 + \operatorname{Sin.} 2 m \alpha \pi^2) \, \Sigma P_m^a \operatorname{Cos.} (m - 2 a) \varphi$$

aut $Q \operatorname{Cos.} 2 m (\acute{\alpha} - \alpha) \pi = \Sigma P_m^a \operatorname{Cos.} (m - 2 a) \varphi$.

Si vero primam per Sin. $2 m \alpha \pi$ et secundam per Cos. $2 m \alpha \pi$ multiplicatas subtractione contraho, fit

$$Q \operatorname{Sin.} 2 m (\acute{\alpha} - \alpha) \pi = \Sigma P_m^a \operatorname{Sin.} (m - 2 a) \varphi \qquad (L.)$$

ubi $\acute{\alpha}$ adhuc definiendum est.

§. 12.

Ad hoc melius perspiciendum petit Poisson summas serierum

$Y = \Sigma P_m^a y^a \operatorname{Cos.} (m - 2 a) \varphi$ et $Z = \Sigma P_m^a y^a \operatorname{Sin.} (m - 2 a) \varphi$

ubi φ inter o et π assumit. Si vero arcus major est $\psi = \varphi + \alpha \pi$, series illae fiunt

$$\Sigma P_m^{a} y^{a} \text{Cos.} (m - 2a)(\varphi + \alpha\pi) = \text{Cos.}\, m\alpha\pi\, Y - \text{Sin}\, m\alpha\pi\, Z$$

$$\Sigma P_m^{a} y^{a} \text{Sin.} (m - 2a)(\varphi + \alpha\pi) = \text{Sin.}\, m\alpha\pi\, Y + \text{Cos.}\, m\alpha\pi\, Z$$

itaque tantummodo Y et Z summandae sunt.

Sed $Y = \Sigma P_m^{a} \dfrac{e^{(m-2a)\varphi i} + e^{-(m-2a)\varphi i}}{2} y^{a}$

$= \frac{1}{2} \Sigma P_m^{a} e^{-2a\varphi i} . y^{a} . e^{m\varphi i} + \frac{1}{2} \Sigma P_m^{a} e^{2a\varphi i} . y^{a} . e^{-m\varphi i}$

$= \frac{1}{2} e^{m\varphi i} \left(1 + y e^{-2\varphi i}\right)^m + \frac{1}{2} e^{-m\varphi i} \left(1 + y e^{2\varphi i}\right)^m.$

$= \frac{1}{2} \left(e^{\varphi i} + y e^{-\varphi i}\right)^m + \frac{1}{2} \left(e^{-\varphi i} + y e^{\varphi i}\right)^m$

Et simili modo mutatur Z in

$$Z = \frac{1}{2i} \left\{ \left(e^{i\varphi} + y e^{-i\varphi}\right)^m - \left(e^{-i\varphi} + y e^{i\varphi}\right)^m \right\}.$$

Est vero $e^{i\varphi} + y e^{-i\varphi} =$

$\frac{1}{2} e^{i\varphi} + \frac{1}{2} e^{-i\varphi} + e^{i\varphi} - \frac{1}{2} e^{-i\varphi} + \frac{1}{2} y e^{-i\varphi} + \frac{1}{2} y e^{+i\varphi} + \frac{1}{2} y e^{-i\varphi} - \frac{1}{2} y e^{i\varphi}$

$= \text{Cos.}\,\varphi + i\, \text{Sin.}\,\varphi + y\, \text{Cos.}\,\varphi - i y\, \text{Sin.}\,\varphi = (1 + y)\, \text{Cos.}\,\varphi + i\,(1 - y)\, \text{Sin.}\,\varphi;$

nam $e^{i\varphi}$ et $y e^{-i\varphi}$ in dimidias partes dissoluta sunt et $\frac{1}{2} e^{-i\varphi}$, $\frac{1}{2} y e^{+i\varphi}$ cum positivo et negativo signo adjecta, invicem se tollunt.

Ponit $R = \sqrt{(1 + y)^2\, \text{Cos.}\,\varphi^2 + (1 - y)^2\, \text{Sin.}\,\varphi^2}$

et $\text{Cos.}\,\vartheta = \dfrac{(1 + y)\, \text{Cos.}\,\varphi}{R}$, $\text{Sin.}\,\vartheta = \dfrac{(1 - y)\, \text{Sin.}\,\varphi}{R}$

ubi R valorem absolutum et ϑ arcum realem significet. Quum itaque

$$e^{i\varphi} + e^{-i\varphi} y = (1 + y)\, \text{Cos.}\,\varphi + i\,(1 - y)\, \text{Sin.}\,\varphi,$$

illis signis adhibitis est

$$e^{i\varphi} + y e^{-i\varphi} = R(\text{Cos.}\,\vartheta + i \text{Sin.}\,\vartheta);\; e^{-i\varphi} + y e^{i\varphi} = R\,(\text{Cos.}\,\vartheta - i\, \text{Sin.}\,\vartheta)$$

Quibus substitutionibus inductis

$$Y = \frac{(e^{i\varphi} + e^{-i\varphi} y)^m + (e^{-i\varphi} + y e^{i\varphi})^m}{2}$$

$$= \frac{R^m (\text{Cos. } m\vartheta + i \text{ Sin. } m\vartheta) + R^m (\text{Cos. } m\vartheta - i \text{ Sin. } m\vartheta)}{2}$$

$$Y = R^m \text{ Cos. } m\vartheta$$

et simili modo $Z = R^m \text{ Sin. } m\vartheta$

ubi R^m semper valorem absolutum significat. At, $y = 1$ posito, series illae transeunt Y in $\Sigma P_m^a \text{ Cos. } (m-2a)\varphi$, Z in $\Sigma P_m^a \text{ Sin. } (m-2a)\varphi$; porro fit $R = \sqrt{2 \text{ Cos. } \varphi^2} = 2 \text{ Cos. } \varphi$; $\text{Cos. } \vartheta = 1$ et $\text{Sin. } \vartheta = o$, itaque $\vartheta = o$, quia minimus arcus esse debet. Sequitur ergo

$$\Sigma P_m^a \text{ Cos. } (m - 2a)\varphi = |2 \text{ Cos. } \varphi|^m \text{ Cos. } m. o = |2 \text{ Cos. } \varphi|^m$$

$$\text{et } \Sigma P_m^a \text{ Sin. } (m - 2a)\varphi = |2 \text{ Cos. } \varphi|^m \text{ Sin. } m. o = o$$

$$\text{Porro } \Sigma P_m^a \text{ Cos. } (m-2a)(\varphi + \alpha\pi) = |2 \text{ Cos. } \varphi|^m \text{ Cos. } m\alpha\pi$$

$$\text{et } \Sigma P_m^a \text{ Sin. } (m-2a)(\varphi + \alpha\pi) = |2 \text{ Cos. } \varphi|^m \text{ Sin. } m\alpha\pi. \qquad M$$

ubi hic, sicuti semper per $|2 \text{ Cos. } \varphi|^m$ valorem realem absolutum potestatis notamus.

§. 13.

Hoc autem non explet, quod Poisson exoptat; quia (6 § 11) unitatis radices desiderentur, α et α' (§ 11 L) adhuc constituenda sint et in seriebus secundum periodicas quantitates procedentibus, indeterminati coefficientes non admovendi sint, nisi alia ratione summa probetur. Itaque aliam solutionem experitur. Ponit (Ferussac Bulletin IV p. 344)

$$y = \text{Cos. } \varphi^m, \qquad \frac{dy}{d\varphi} \text{ Cos. } \varphi + my \text{ Sin. } \varphi = o$$

quibus respondere potest:

$$y = A + B \text{ Cos. } \varphi + C \text{ Cos. } 2\varphi + D \text{ Cos. } 3\varphi \ldots + N \text{ Cos. } n\varphi$$

$$-\frac{dy}{d\varphi} = B \text{ Sin. } \varphi + 2 C \text{ Sin. } 2\varphi + 3 D \text{ Sin. } 3\varphi \ldots + n N \text{ Sin. } n\varphi.$$

Quibus valoribus substitutis, fit

$$\begin{aligned} o = {} & m A \text{ Sin. } \varphi + m B \text{ Cos. } \varphi + m C \text{ Cos. } 2\varphi \text{ Sin. } \varphi \\ & + m D \text{ Cos. } 3\varphi \text{ Sin. } \varphi \ldots + m N \text{ Cos. } n\varphi \text{ Sin. } \varphi \ldots \\ & - B \text{ Cos. } \varphi \text{ Sin. } \varphi - 2 C \text{ Cos. } \varphi \text{ Sin. } 2\varphi \\ & - 3 D \text{ Cos. } \varphi \text{ Sin. } 3\varphi \ldots - n N \text{ Cos. } \varphi \text{ Sin. } n\varphi \ldots \end{aligned}$$

Secundum formulas jam § 4 adhibitas series mutari potest in

$$o = m\,A\,\mathrm{Sin.}\,\varphi + \tfrac{1}{2}\,m\,B\,\mathrm{Sin.}\,2\,\varphi$$
$$- \tfrac{1}{2}\,m\,C\,\mathrm{Sin.}\,\varphi - \tfrac{1}{2}\,m\,D\,\mathrm{Sin.}\,2\,\varphi + \tfrac{1}{2}\,m\,C\,\mathrm{Sin.}\,3\,\varphi + \tfrac{1}{2}\,m\,D\,\mathrm{Sin.}\,4\,\varphi$$
$$- \tfrac{1}{2}\,m\,E\,\mathrm{Sin.}\,3\,\varphi - \tfrac{1}{2}\,m\,F\,\mathrm{Sin.}\,4\,\varphi$$
$$- C\,\mathrm{Sin.}\,\varphi - \tfrac{1}{2}\,B\,\mathrm{Sin.}\,2\,\varphi \quad - C\,\mathrm{Sin.}\,3\,\varphi$$
$$- \tfrac{3}{2}\,D\,\mathrm{Sin.}\,2\,\varphi \quad - 2\,E\,\mathrm{Sin.}\,3\,\varphi - \tfrac{3}{2}\,D\,\mathrm{Sin.}\,4\,\varphi \ldots$$
$$- \tfrac{5}{2}\,F\,\mathrm{Sin.}\,4\,\varphi \ldots$$

Terminis eodem sinu multiplicatis, pro se evanescentibus oriuntur aequationes, ex quibus definiuntur $C = \frac{1}{2}\,A\frac{m}{m+2}$; $D = B\frac{m-1}{m+3}$;

$$E = C\frac{m-2}{m+4} = A\frac{m.\,m-2}{m+2.\,m+4};\quad F = D\frac{m-3}{m+5} = B\frac{m-1.\,m-3}{m+3.\,m+5}.$$

ubi A et B indeterminatae quantitates remanent; et fit $y = A\,Q + B\,R$, si

$$Q = \tfrac{1}{2} + \frac{m}{m+2}\mathrm{Cos.}\,2\,\varphi + \frac{m.\,m-2}{m+2.\,m+4}\,\mathrm{Cos.}\,4\,\varphi \ldots + \frac{m.\,m-2\ldots m-2\,(a-1)}{m+2\ldots\ldots m+2\,a}.\ \mathrm{Cos.}\,2\,a\,\varphi$$
$$R = \mathrm{Cos.}\,\varphi + \frac{m-1}{m+3}\,\mathrm{Cos.}\,3\,\varphi + \frac{m-1.\,m-3}{m+3.\,m+5}\,\mathrm{Cos.}\,5\,\varphi \ldots \frac{m-1.\,m-3\ldots m-(2a-1)}{m+3.\,m+5\ldots(m+2a+1)}\mathrm{Cos.}(2a+1)\,\varphi$$

ubi iterum duae arbitrariae constantes quantitates ex una aequatione differentiali primi ordinis prodire videntur; sed inter se cohaereant oportet. Fiat $Q = \lambda$ et $R = \mu$, si $\varphi = o$ ponitur; ergo $(1)^m = \lambda\,A + \mu\,B$. Sed $\varphi = \pi$ posito Q eundem valorem servat, sed R fit $= -\mu$, quia cosinus imparis numeri $\pi = -1$ est; itaque $(\mathrm{Cos.}\,\pi)^m = (-1)^m = \lambda\,A - \mu\,B$

Erat autem $(+1)^m = \lambda\,A + \mu\,B$ ergo

$$2\,A\,\lambda = (-1)^m + (+1)^m;\ \text{et}\ 2\,B\,\mu = (+1)^m - (-1)^m$$

ex quibus sequitur

$$y = \frac{1}{2\lambda}\left\{(+1)^m + (-1)^m\right\}Q + \frac{1}{2\mu}\left\{(+1)^m - (-1)^m\right\}R$$

Si m numerus integer et impar est, fit $y = \frac{1}{\mu}\,R$, si par est, $y = \frac{1}{\lambda}\,Q$; quo casu etiam demonstrari potest, $R = o$. — Sed utique altera quantitatum R aut Q elidi potest, nam $\pm\frac{Q}{\lambda} = \frac{R}{\mu}$, quod Poisson, indeterminatorum coefficientium methodo non satis confidens, probat in formula alio loco (Journal pour l'école polytechnique cah. 19me p. 434) demonstrata nixus.

§. 14.

Poinsot (Recherches sur l'analyse des sections angulaires 1825. § 5.) methodos antea adhibitas emendat. Viam, quam Lagrange ingressus erat, sequitur usque ad constantis definitionem $y = \mathrm{Cos.}\,\varphi^m = A\,\Sigma\, P_m^a \mathrm{Cos.}(m-2a)\varphi$. Sed $\varphi = o$ posito $y = (1)^m$ non semper $= 1$ fit, sed

$= \mathrm{Cos.}\ 2\,m\,\alpha\,\pi + i\ \mathrm{Sin.}\ 2\,m\,\alpha\,\pi$. At $\varphi = \pi$ posito,

$y = (-1)^m = \mathrm{Cos.}\ (2\,\alpha + 1)\,m\,\pi + i\ \mathrm{Sin.}\ (2\,\alpha + 1)\,m\,\pi$. Itaque habemus ad constantem definiendam

$(\underline{+}1)^m = A\,2^m\,\mathrm{Cos.\,m\,arc.}(\mathrm{Cos.}\ \underline{+}\,1)$; nam sicuti π ita etiam omnino $\alpha\,\pi$ sumere possumus ad constantem definiendam; ergo

$$A = \frac{(+1)^m}{2^m\,.\ \mathrm{Cos.\ m\,arc.}\ (\mathrm{Cos.} = +1)}\ \text{aut} = \frac{(-1)^m}{2^m\ \mathrm{Cos.\,m\,arc.}\ (\mathrm{Cos.} = -1)}$$

Quo valore in superiorem formulam posito fit

$$(2\ \mathrm{Cos.}\ \varphi)^m = \frac{(\underline{+}1)^m}{\mathrm{Cos.m\,arc.}(\mathrm{Cos.}\underline{+}1)}\ \Sigma\, P_m^a\ \mathrm{Cos.}\ (m - 2\,a)\,\varphi. \qquad \text{N}$$

Si m numerus integer est, fit $A = \frac{1}{2^m}$; sed si m est numerus fractus $\frac{r}{n}$,

A praebet n varios valores propter radices unitatis $(+1)^{\frac{r}{n}}$; sed quia in serie $\Sigma\, P_m^a\ \mathrm{Cos.}\ (m - 2\,a)\,\varphi$ etiam pro φ poni potest $\varphi + 2\,\alpha\,\pi$, etiam ex ea n varii valores efficiuntur; quapropter n. n valores exhiberi videntur. Quia hoc locum habere non potest, arcus in denominatore eodem modo mutandas est ac in serie, ita ut quotus constans maneat. Ergo si φ arcum significat inter $-\frac{1}{2}\,\pi$ et $+\frac{1}{2}\,\pi$, fit

$$(2\ \mathrm{Cos.}\ \varphi)^m = \frac{(+1)^m}{\mathrm{Cos.}\ m\,\alpha\,\pi}\ \Sigma\, P_m^a\ \mathrm{Cos.}\ (m - 2\,a)\ (\varphi + \alpha\,\pi).$$

Exemplis nonnullis $\varphi = \frac{\pi}{4}, = \frac{3\pi}{4}$ Poinsot demonstrat, formulam suam veram summam praebere, nempe

$(2\ \mathrm{Cos.}\ \frac{1}{4}\,\pi)^m = (1)^m|\sqrt{2}|^m$ et $(2\ \mathrm{Cos.}\ \frac{3}{4}\,\pi)^m = (-1)^m\,|\sqrt{2}|^m$

Nam si cos. φ negativus est, pro $(+1)^m$ ponendum est $(-1)^m$. -- Ab Euleri formula illa (N) modo factore $\frac{(+1)^m}{\text{Cos. m (arc.Cos.} +1)}$ discrepat. Sed si φ inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ jacet, Cos. φ positivus est, ergo $(+1)^m$ sumendum et arcus, cujus Cos. $= 1$, $= 0$, itaque Cos. m o $= 1$, quamobrem Euleri aequatio semper recta est pro quocunque exponente, si φ inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$; nam $(+1)^m$ semper habet unum valorem $= +1$. Similis autem limitatio aliis etiam formulis aptanda est, ut $(1 + x)^m = \Sigma P^a_m x^a$ tantummodo valet, si x inter $+1$ et -1, et si m quantitas positiva est.

§. 15.

Praeterea formulam a Poisson tractatam (§ 11)
$|2\,\text{Cos.}\,\varphi|^m \{\text{Cos.}\,2m\acute{\alpha}\pi + i\,\text{Sin.}\,2m\acute{\alpha}\pi\} = \Sigma P^a_m \text{Cos.}\,(m-2a)(\varphi+2\alpha\pi) + i\,\Sigma P^a_m \text{Sin.}\,(m-2a)(\varphi+2\alpha\pi)$ iterum contemplatur, ut valorem α et $\acute{\alpha}$ constituat. Ad quem finem ponit $\varphi = o$ et eo invenit $\acute{\alpha} = \alpha$, quia et realis et imaginaria pars pro se evanescere debet. Inde sequuntur aequationes

$$|2\,\text{Cos.}\,\varphi|^m \text{Cos.}\,2m\alpha\pi = \Sigma P^a_m \text{Cos.}\,(m-2a)(\varphi+2\alpha\pi)$$
$$|2\,\text{Cos.}\,\varphi|^m \text{Sin.}\,2m\alpha\pi = \Sigma P^a_m \text{Sin.}\,(m-2a)(\varphi+2\alpha\pi).$$

quam ob causam $\frac{\text{Cos.}\,2m\alpha\pi}{\text{Sin.}\,2m\alpha\pi} = \frac{\Sigma P^a_m \text{Cos.}\,(m-2a)(\varphi+2\alpha\pi)}{\Sigma P^a_m \text{Sin.}\,(m-2a)(\varphi+2\alpha\pi)}$

Quod manifeste cohaerentiam quantitatum constantium aperit, quibus utraque series multiplicanda est et de quibus Lacroix (cf. § 10) egit. — Si vero Cos. φ negativus est, ergo $(2\,\text{Cos.}\,\varphi)^m = (-1)^m\,|2\,\text{Cos.}\,\varphi|^m$, fit

$$|2\,\text{Cos.}\,\varphi|^m \{\text{Cos.}\,(2\alpha+1)m\pi + i\,\text{Sin.}\,(2\alpha+1)m\pi\}$$
$$= \Sigma P^a_m \text{Cos.}\,(m-2a)(\varphi+2\alpha\pi) + i\,\Sigma P^a_m \text{Sin.}\,(m-2a)(\varphi+2\alpha\pi)$$

ubi φ inter $\frac{1}{2}\pi$ et $\frac{3}{2}\pi$ situm est. At partibus reali et imaginaria invicem sibi paribus positis, fit

$$|2\,\text{Cos.}\,\varphi|^m = \frac{\Sigma P^a_m \text{Cos.}\,(m-2a)(\varphi+2\alpha\pi)}{\text{Cos.}\,m(2\alpha+1)\pi}$$

$$\text{et } |2\,\text{Cos.}\,\varphi|^m = \frac{\Sigma\, P_m^a\, \text{Sin.}\,(m-2a)(\varphi + 2\alpha\pi)}{\text{Sin.}\, m\,(2\alpha+1)\,\pi}$$

$$\frac{\text{Cos.}\, m\,(2\alpha+1)\,\pi}{\text{Sin.}\, m\,(2\alpha+1)\,\pi} = \frac{\Sigma\, P_m^a\, \text{Cos.}\,(m-2a)(\varphi+2\alpha\pi)}{\Sigma\, P_m^a\, \text{Sin.}\,(m-2a)(\varphi+2\alpha\pi)}$$

quae aequationes manifestant, ex Euleri formula omnes valores eo exhiberi non posse, ut pro φ ponatur $\varphi + 2\alpha\pi$; nam si illic pro α ponimus omnes numeros integros et $m = \frac{r}{n}$, n quidem varios valores habemus, sed omnes reales; at duo modo reales esse possunt; ergo series ita sunt dividendae, ut quotus constans sit.

§. 16.

Ohm in prima sectione libri sui: Aufsätze aus dem Gebiete der höheren Mathematik 1823, inscripti, putat causam erroris, in quem Euler lapsus sit, in eo sitam esse, quod (1 et 2 § 3) varii potestatis valores sint. Sed ut jam Poinsot, ita paulo plenius infra demonstrare conabor, Euleri methodo unum et quidem realem valorem inveniri. Rationem autem a Lagrange adhibitam vituperat, quia coefficientes indefiniti modo admovendi sint, si alia via seriei forma certior facta sit; de quo § 28 agam. Ohm ipse hunc calculum init:

$$(2\,\text{Cos.}\,\varphi)^m = \left(e^{i\varphi} + e^{-i\varphi}\right)^m = \Sigma\, P_m^a\, e^{(m-2a)\,i\varphi}$$

quia $2\,\text{Cos.}\,\varphi = e^{i\varphi} + e^{-i\varphi}$; deinde secundum

$e^{i\chi} = \text{Cos.}\,(\chi + 2\alpha\pi) + i\,\text{Sin.}\,(\chi + 2\alpha\pi)$ mutat $\Sigma\, P_m^a\, e^{(m-2a)\,i\varphi}$

$$= \Sigma\, P_m^a \left\{\text{Cos.}\,(m-2a)(\varphi+2\alpha\pi) + i\,\text{Sin.}\,(m-2a)(\varphi+2\alpha\pi)\right\}$$

$$(2\,\text{Cos.}\,\varphi)^m = \text{Cos.}\,2m\alpha\pi\, \Sigma\, P_m^a\, \text{Cos.}\,(m-2a)\,\varphi - \text{Sin.}\,(2m\alpha\pi)\, \Sigma\, P_m^a\, \text{Sin.}\,(m-2a)\,\varphi$$
$$+\, i \left\{\text{Sin.}\,2m\alpha\pi\, \Sigma\, P_m^a\, \text{Cos.}\,(m-2a)\,\varphi + \text{Cos.}\,2m\alpha\pi\, \Sigma\, P_m^a\, \text{Sin.}\,(m-2a)\,\varphi\right.$$

In quarta sectione ejusdem libri § 67 Ohm proponit series summandas

$\Sigma\, P_z^a\, \text{Cos.}\, a\gamma = R$ et $\Sigma\, P_z^a\, \text{Sin.}\, a\gamma = Q$

$$R = \Sigma\, P_z^a\, \frac{e^{ia\gamma} + e^{-ia\gamma}}{2} = \frac{\left(1 + e^{i\gamma}\right)^z + \left(1 + e^{-i\gamma}\right)^z}{2}$$

$$= \frac{(1 + \text{Cos.}\,\gamma + i\,\text{Sin.}\,\gamma)^z + (1 + \text{Cos.}\,\gamma - i\,\text{Sin.}\,\gamma)^z}{2}$$

Sed $1 + \text{Cos.}\,\gamma = 2\,\text{Cos.}\,\tfrac{1}{2}\gamma^2$ et $\text{Sin.}\,\gamma = 2\,\text{Cos.}\,\tfrac{1}{2}\gamma\,\text{Sin.}\,\tfrac{1}{2}\gamma$, itaque $(1 + \text{Cos.}\,\gamma \pm i\,\text{Sin.}\,\gamma) = 2\,\text{Cos.}\,\tfrac{1}{2}\gamma\,(\text{Cos.}\,\tfrac{1}{2}\gamma \pm i\,\text{Sin.}\,\tfrac{1}{2}\gamma)$; ergo

$$R = \frac{(2\,\text{Cos.}\,\tfrac{1}{2}\gamma)^z}{2}\left\{(\text{Cos.}\,\tfrac{1}{2}\gamma + i\,\text{Sin.}\,\tfrac{1}{2}\gamma)^z + (\text{Cos.}\,\tfrac{1}{2}\gamma - i\,\text{Sin.}\,\tfrac{1}{2}\gamma)^z\right\}$$

$$= \frac{(2\,\text{Cos.}\,\tfrac{1}{2}\gamma)^z}{2}\Big\{\text{Cos.}\,(\tfrac{1}{2}\gamma + 2\,\alpha\,\pi)\,z + i\,\text{Sin.}\,(\tfrac{1}{2}\gamma + 2\,\alpha\,\pi)\,z$$

$$+ \text{Cos.}\,(\tfrac{1}{2}\gamma + 2\,\alpha'\,\pi)\,z - i\,\text{Sin.}\,(\tfrac{1}{2}\gamma + 2\,\alpha'\,\pi)\,z\Big\}$$

Nam α et α' adhuc definienda inducit, quia fieri possit, ut varii valores in calculum vocandi sint. Si $\text{Cos.}\,\tfrac{1}{2}\gamma$ positivus est, imaginariae partes pro se evanescere debent, quia tunc et R et $(2\,\text{Cos.}\,\tfrac{1}{2}\gamma)^z$ reales quantitates sunt. Fit autem $\text{Sin.}\,(\tfrac{1}{2}\gamma + 2\,\alpha\,\pi)\,z = \text{Sin.}\,(\tfrac{1}{2}\gamma + 2\,\alpha'\,\pi)\,z$, si $\alpha' = \alpha$ sumitur. Quo facto sequitur

$$R = \frac{(2\,\text{Cos.}\,\tfrac{1}{2}\gamma)^z}{2}.\,2\,\text{Cos.}\,(\tfrac{1}{2}\gamma + 2\,\alpha\,\pi)\,z =$$

$$(2\,\text{Cos.}\,\tfrac{1}{2}\gamma)^z\;\text{Cos.}\,(\tfrac{1}{2}\gamma + 2\,\alpha\,\pi)\,z = \Sigma\,P_z^{\mathfrak{a}}\,\text{Cos.}\,\mathfrak{a}\,\gamma,$$

si $\tfrac{1}{2}\gamma$ inter $2\,\alpha \mp \tfrac{1}{2}\pi$ jacet. — Eodem modo pertractanda est series Q, quae fit

$$\Sigma\,P_z^{\mathfrak{a}}\,\text{Sin.}\,\mathfrak{a}\,\gamma = (2\,\text{Cos.}\,\tfrac{1}{2}\gamma)^z\;\text{Sin.}\,(\tfrac{1}{2}\gamma + 2\,\alpha\,\pi)\,z$$

Posito ergo $\tfrac{1}{2}\gamma = \varphi$ et $z = m$, fiunt series illae

$$\Sigma\,P_m^{\mathfrak{a}}\,\text{Cos.}\,\mathfrak{a}\,2\,\varphi = (2\,\text{Cos.}\,\varphi)^m\;\text{Cos.}\,(\varphi + 2\,\alpha\,\pi)\,m$$

$$\Sigma\,P_m^{\mathfrak{a}}\,\text{Sin.}\,2\,\mathfrak{a}\,\varphi = (2\,\text{Cos.}\,\varphi)^m\;\text{Sin.}\,(\varphi + 2\,\alpha\,\pi)\,m;$$

quibus potestas secundum crescentes arcus multiplices evolvuntur. Ex iis autem alias formulas deducit:

$$\text{Cos.}\,m\,\varphi\,\Sigma\,P_m^{\mathfrak{a}}\,\text{Cos.}\,2\,\mathfrak{a}\,\varphi = (2\,\text{Cos.}\,\varphi)^m\;\text{Cos.}\,m\,\varphi\;\text{Cos.}\,m\,(\varphi + 2\,\alpha\,\pi)$$

$$\text{Sin.}\,m\,\varphi\,\Sigma\,P_m^{\mathfrak{a}}\,\text{Sin.}\,2\,\mathfrak{a}\,\varphi = (2\,\text{Cos.}\,\varphi)^m\;\text{Sin.}\,m\,\varphi\;\text{Sin.}\,m\,(\varphi + 2\,\alpha\,\pi)$$

additione conjunctae $\Sigma\,P_m^{\mathfrak{a}}\,\text{Cos.}\,(m - 2\,\mathfrak{a})\,\varphi = (2\,\text{Cos.}\,\varphi)^m\;\text{Cos.}\,2\,m\,\alpha\,\pi.$

et $\text{Sin.}\,m\,\varphi\,\Sigma\,P_m^{\mathfrak{a}}\,\text{Cos.}\,2\,\mathfrak{a}\,\varphi = (2\,\text{Cos.}\,\varphi)^m\;\text{Sin.}\,m\,\varphi\;\text{Cos.}\,m\,(\varphi + 2\,\alpha\,\pi)$

$$\text{Cos.}\,m\,\varphi\,\Sigma\,P_m^{\mathfrak{a}}\,\text{Sin.}\,2\,\mathfrak{a}\,\varphi = (2\,\text{Cos.}\,\varphi)^m\;\text{Cos.}\,m\,\varphi\;\text{Sin.}\,m\,(\varphi + 2\,\alpha\,\pi)$$

subtractione $\Sigma\,P_m^{\mathfrak{a}}\,\text{Sin.}\,(m - 2\,\mathfrak{a})\,\varphi = (2\,\text{Cos.}\,\varphi)^m\;\text{Sin.}\,2\,m\,\alpha\,\pi.$

ubi Cos. φ positivus est et φ inter $(2\alpha \mp \frac{1}{2})\pi$. — Si vero Cos. ψ negativus est et ψ inter $(2\alpha + \frac{1}{2})\pi$ et $(2\alpha + \frac{3}{2})\pi$ jacet, ergo $\psi = \varphi + \pi$, et $(\text{Cos.}\,\psi)^m = |\text{Cos.}\,\varphi|^m (\text{Cos. } m\pi + i \text{ Sin. } m\pi)$ ex antecedentibus deduci possunt aequationes

$$\Sigma\, P_m^a \,\text{Sin.}(m-2a)\left\{(2\alpha+1)\pi+\varphi\right\} = (2\,\text{Cos.}\,\psi)^m\,\text{Sin.}(2\alpha+1)m\pi$$

$$\text{et } \Sigma\, P_m^a \,\text{Cos.}(m-2a)\left\{(2\alpha+1)\pi+\varphi\right\} = (2\,\text{Cos.}\,\psi)^m\,\text{Cos.}(2\alpha+1)m\pi.$$

§. 17.

Crelle (Versuch einer allgemeinen Theorie der analytischen Facultäten) methodo ab ea § 11. prolata non valde discrepante pervenit ad

$$(2\,\text{Cos.}\,\varphi)^m = \Sigma\, P_m^a \left\{\text{Cos.}(m-2a)(\varphi+2\alpha\pi) + i\,\text{Sin.}(m-2a)(\varphi+2\alpha\pi)\right\} = |2\text{Cos.}\,\varphi|^m (\pm 1)^m.$$

et enumerat omnes casus, quibus $(+1)^m$ et $(-1)^m$ valores reales et quibus omnino imaginarios aut formae $A + iB$ habeant. $(+1)^m$ et $(-1)^m$ semper unum habent valorem realem, si m est numerus integer, nempe $(+1)^m$ semper $= +1$; $(-1)^m = +1$, m numero pari, $= -1$, m impari. — Si m numerus fractus est, minimis numeris expressus $= \frac{r}{n}$, $(+1)^{\frac{r}{n}}$ semper valorem realem habet, nempe $(+1)^{\frac{r}{2n+1}} = +1$ et $(+1)^{\frac{r}{2n}} = \pm 1$. Valor omnino imaginarius exstat $(+1)^{\frac{r}{n}} = \pm i$, si denominator quadripartitus est. Sed τῷ $(-1)^{\frac{r}{n}}$ realis valor non respondet, si denominator exponentis fracti par est; sed si denominator impar est, fit $(-1)^{\frac{r}{2n+1}} = +1$ pari, $= -1$, impari nominatore. Valores autem omnino imaginarii modo existunt, si denominator par est et formae $4(2\beta+1) - 2$, ubi β quemcunque numerum integrum significare potest. Si m est quantitas positiva irrationalis, transcendens aut imaginaria, $(+1)^m$ tantummodo unum valorem realem habet $= +1$. Ceteris omnibus casibus $(+1)^m$ et $(-1)^m$ sunt formae $A + iB$. Quum vero

$$(2\,\mathrm{Cos.}\,\varphi)^m = \mathrm{Cos.}\,2\,m\,\alpha\,\pi\,\Sigma\,P_m^a\,\mathrm{Cos.}\,(m-2\,a)\,\varphi$$
$$-\,\mathrm{Sin.}\,2\,m\,\alpha\,\pi\,\Sigma\,P_m^a\,\mathrm{Sin.}\,(m-2\,a)\,\varphi$$
$$+\,i\,\{\,\mathrm{Sin.}\,2\,m\,\alpha\,\pi\,\Sigma\,P_m^a\,\mathrm{Cos.}\,(m-2\,a)\,\varphi$$
$$+\,\mathrm{Cos.}\,(2m\,\alpha\,\pi\,\Sigma\,P_m^a\,\mathrm{Sin.}\,(m-2\,a)\,\varphi\}$$

concludit Crelle, iis casibus, quibus $(+1)^m$ et $(-1)^m$ reales valores habeant, priore parte potestatem exprimendam esse; quibus autem valores utique imaginarii existant, secunda parte; et affirmat, Euleri formulam tantummodo valere, si exponens m numerus integer sit. Generalis aequatio ei est $(2\,\mathrm{Cos}\;\varphi)^m = \pm\sqrt{\{\Sigma\,P_m^a\,\mathrm{Cos.}\,(m-2\,a\,\varphi)^2 + \Sigma\,P_m^a\,\mathrm{Sin.}\,(m-2\,a)\,\varphi^2\}}$

§. 18.

Eandem rem Olivier (Crelle Journal I p. 16) tractat; e formula

$$(2\,\mathrm{Cos.}\,\varphi)^m = \Sigma\,P_m^a\,\{\mathrm{Cos.}\,(m-2\,a)\,(\varphi+2\,\alpha\,\pi) + i\,\mathrm{Sin.}\,(m-2\,a)\,(\varphi+2\,\alpha\,\pi)\}$$

profectus, studet certis valoribus τῶν φ et m singularem valorem pro $(2\,\mathrm{Cos.}\,\varphi)^m$ invenire. Quum vero $(2\,\mathrm{Cos.}\,\varphi)^m = |2\mathrm{Cos.}\,\varphi|^m\,(+1)^m$ sit, constituit valores reales et imaginarios τοῦ $(+1)^m$ ita ut Crelle (§ 17). Sed ut $(-1)^m = \mathrm{Cos.}\,m\,(2\,\alpha+1)\,\pi + i\,\mathrm{Sin.}\,m\,(2\,\alpha+1)\,\pi$ realem valorem habeat, oportere contendit $m\,(2\,\alpha+1) = \lambda$ et quidem λ numerum imparem; si vero $m = \frac{r}{2n+1}$ r par esse potest et $\frac{2\alpha+1}{2n+1}$ aequum numero integro, itaque λ par esse potest. Deinde $(-1)^m$ valores omnino imaginarios habere negat; quorum conditionem Crelle jam patefecit.

Recte Olivier contendit, pro positivo cosinu φ semper imaginariam partem evanescere debere, ergo

$$o = \Sigma\,P_m^a\,\mathrm{Sin.}\,(m-2\,a)\,(\varphi+2\,\alpha\,\pi)$$
$$= \mathrm{Cos.}\,2\,m\,\alpha\,\pi\,\Sigma\,P_m^a\,\mathrm{Sin.}\,(m-2\,a)\,\varphi + \mathrm{Sin.}\,2\,m\,\alpha\,\pi\,\Sigma\,P_m^a\,\mathrm{Cos.}(m-2a)\,\varphi.$$

et ostensurus est, $\mathrm{Cos.}\,2\,m\,\alpha\,\pi$ et $\mathrm{Sin.}\,2\,m\,\alpha\,\pi$ eundem valorem servare, donec $\mathrm{Cos.}\,\varphi$ idem signum retineat. Ad quod demonstrandum ponit, $\alpha = p$ respondere arcui $a-b$, et $\alpha = p+q$ arcui $a+c$, qui arcus ita sumendi sunt, ut $a-b$ et $a+c$ inter $(\tau-\frac{1}{2})\,\pi$ et $(\tau+\frac{1}{2})\,\pi$ cadant. Ergo his signis

$$o = \Sigma\,P_m^a\,\mathrm{Sin.}\,(m-2\,a)\,(a-b+2\,p\,\pi)$$
$$\text{et}\quad o = \Sigma\,P_m^a\,\mathrm{Sin.}\,(m-2\,a)\,(a+c+2\,\overline{p+q}\,\pi)$$

simul existere debent. Ponit autem, quod sine dubio licet, $b = o$, et $c = o$, quia aequatio etiam pro arcu a valet, inter illos fines sito; sed ponendo $b = o$ et $c = o$ valores p et $p + q$ non amplius diversi sunt, quod eum fugit, sed $q = o$; formulae inde deductae bene se habent, $q = o$ posito, sed nihil probant. — Postea petit, ut in factoribus Cos. $2\, m\, \alpha\, \pi$ et Sin. $2\, m\, \alpha\, \pi$ numerum α constituat, quod necessarium non fuisset, si unitatis radices bene exposuisset, quia eaedem conditiones valent. Ex falsa supra notata methodo constantium determinandarum argumentatur

$|2 \text{ Cos. } \varphi|^m = \Sigma P_m^a \text{ Cos. } (m - 2a)(\varphi - 2\alpha\pi)$ universe pro positivo Cos. φ valere.

§. 19.

Addimus methodum, qua Cauchy (Exercices de mathématiques Paris 1826) in dissertatione: „sur l'analyse des sections angulaires“ ad nostram quaestionem solvendam utitur. — Brevitatis causa ponit

$$u = \Sigma P_m^a \text{ Cos. } (m - 2a)\varphi \text{ et } v = \Sigma P_m^a \text{ Sin. } (m - 2a)\varphi$$

$$U = \Sigma P_m^a y^a \text{ Cos. } (m - 2a)\varphi \text{ et } V = \Sigma P_m^a y^a \text{ Sin. } (m - 2a)\varphi$$

$$\text{ergo } U + iV = \Sigma P_m^a y^a \left\{\text{Cos. } (m - 2a)\varphi + i \text{ Sin. } (m - 2a)\varphi\right\}$$

$$= \Sigma P_m^a y^a e^{(m-2a)i\varphi} = \Sigma P_m^a y^a e^{mi\varphi} . e^{-2ai\varphi}$$

$U + iV = e^{mi\varphi} \Sigma P_m^a y^a e^{-2ai\varphi} = e^{mi\varphi} (1 + y e^{2\varphi i})^m$. Ut haec series convergat, y inter -1 et $+1$ sumendum est; et $y = 1$ posito transit $U + iV$ in $u + iv = e^{mi\varphi} (1 + e^{2\varphi i})^m$. Si vero $\varphi = \vartheta \pm \alpha\pi$, ubi ϑ inter $+\frac{1}{2}\pi$ et $-\frac{1}{2}\pi$ jacet, ponit Cauchy $e^{-2\varphi i} = e^{-2\vartheta i}$, quia $e^{-2\varphi i} =$ (Cos. 2φ — i Sin. 2φ);

$$\text{sed Cos. } 2\varphi = \text{Cos. } 2(\vartheta \pm \alpha\pi) = \text{Cos. } 2\vartheta$$

$$\text{et Sin. } 2\varphi = \text{Sin. } 2(\vartheta \pm \alpha\pi) = \text{Sin. } 2\vartheta$$

$$\text{ergo } e^{-2\varphi i} = \text{Cos. } 2\vartheta - i \text{ Sin. } 2\vartheta = e^{-2\vartheta i}$$ cf. §. 24.

Itaque $u + iv = e^{m\varphi i} (1 - e^{-2\varphi i})^m$ transit in

$$e^{m(\vartheta \dagger \alpha\pi)i} (1 + e^{-2\vartheta i})^m = e^{mi\vartheta} e^{m\alpha\pi i} (1 + e^{-2\vartheta i})^m$$

si potestatem cum $e^{mi\vartheta}$ multiplicamus, fit

$$u + iv = e^{m\alpha\pi i}\left(e^{i\vartheta} + e^{-i\vartheta}\right)^m$$

$$= (\text{Cos. } m\alpha\pi + i \text{ Sin. } m\alpha\pi)(2 \text{ Cos. } \vartheta)^m$$

Reali et imaginaria partibus invicem sibi paribus positis efficitur

$$u = \Sigma P_m^a \text{ Cos. } (m - 2a)\varphi = [2 \text{ Cos. } \vartheta]^m \text{ Cos. } m\alpha\pi$$

$$v = \Sigma P_m^a \text{ Sin. } (m - 2a)\varphi = [2 \text{ Cos. } \vartheta]^m \text{ Sin. } m\alpha\pi.$$

Summae illis, quas Poisson invenit, similes.

§. 20.

Pagani (Ferussac Bulletin V. 1826. No. 6) in universum quaerit, quomodo, si $Z^{\frac{1}{n}} = A + iB$ et Q valor realis est, A et B per Q exprimi possint. Est vero

$$A + iB = Q\left\{\text{Cos. } \frac{2\alpha}{n}\pi + i \text{ Sin. } \frac{2\alpha}{n}\pi\right\},$$ si Z quantitas positiva est, et

$$A + iB = Q\left\{\text{Cos. } \frac{2\alpha + 1}{n}\pi + i \text{ Sin. } \frac{2\alpha+1}{n}\pi\right\}$$

si Z negativa quantitas est. Ex quibus colligitur

$$A = Q \text{ Cos. } \frac{2\alpha}{n}\pi \quad \text{aut} = Q \text{ Cos. } \frac{2\alpha+1}{n}\pi.$$

$$B = Q \text{ Sin. } \frac{2\alpha}{n}\pi \quad \text{aut} = Q \text{ Sin. } \frac{2\alpha+1}{n}\pi$$

denique $\frac{B}{A} = \text{tang. } \frac{2\alpha}{n}\pi \quad \text{aut} = \text{tang. } \frac{2\alpha+1}{n}\pi.$

$A^2 + B^2 = Q^2\left\{\text{Cos. } \frac{2\alpha}{n}\pi^2 + \text{Sin. } \frac{2\alpha}{n}\pi^2\right\}$, itaque $Q = \sqrt{A^2 + B^2}$

qui valor etiam in illis aequationibus substitui potest. — Illis pro A et B inventis valoribus adhibitis $Z^{\frac{1}{n}}$ variis modis exprimi potest:

$$Z^{\frac{1}{n}} = \frac{A}{\text{Cos. } \frac{2\alpha}{n}\pi}\left\{\text{Cos. } \frac{2\alpha}{n}\pi + i \text{ Sin. } \frac{2\alpha}{n}\pi\right\}$$

$$\text{aut } Z^{\frac{1}{n}} = \frac{A}{\text{Cos.} \frac{2\acute{\alpha}+1}{n}\pi} \left\{ \text{Cos.} \frac{2\alpha+1}{n}\pi + i \, \text{Sin.} \frac{2\alpha+1}{n}\pi \right\}$$

$$Z^{\frac{1}{n}} = \frac{B}{\text{Sin.} \frac{2\acute{\alpha}}{n}\pi} \left\{ \text{Cos.} \frac{2\alpha}{n}\pi + i \, \text{Sin.} \frac{2\alpha}{n}\pi \right\}$$

$$\text{aut} = \frac{B}{\text{Sin.} \frac{2\acute{\alpha}+1}{n}\pi} \left\{ \text{Cos.} \frac{2\alpha+1}{n}\pi + i \, \text{Sin.} \frac{2\alpha+1}{n}\pi \right\}$$

ubi $\acute{\alpha}$ adhuc definiendum numerum integrum significat, α autem omnes usque ad $n - 1$. Expressio omnes casus complectens est

$$Z^{\frac{1}{n}} = \sqrt{A^2 + B^2} \, (\pm 1)^{\frac{1}{n}}.$$

Quibus expositis transit ad

$$(2 \text{ Cos. } \varphi)^{\frac{r}{n}} = \sqrt[n]{(2 \text{ Cos. } \varphi)^r} = \overset{a}{\Sigma} P^{\frac{r}{n}} \text{ Cos. } (\tfrac{r}{n} - 2a)\varphi$$

$$+ i \overset{a}{\Sigma} P^{\frac{r}{n}} \text{ Sin. } (\tfrac{r}{n} - 2a)\varphi, \text{ ubi } Z = (2 \text{ Cos. } \varphi)^r.$$

$$\text{Itaque } (2 \text{ Cos. } \varphi)^{\frac{r}{n}} = \frac{\overset{a}{\Sigma} P^{\frac{r}{n}} \text{ Cos. } \left(\frac{r}{n} - 2a\right)\varphi}{\text{Cos.} \frac{2\acute{\alpha}}{n}\pi} (\pm 1)^{\frac{1}{n}}$$

$$= \frac{\overset{a}{\Sigma} P^{\frac{r}{n}} \text{ Sin. } \left(\frac{r}{n} - 2a\right)\varphi}{\text{Sin.} \frac{2\acute{\alpha}}{n}\pi} (\pm 1)^{\frac{1}{n}}$$

$$\text{Tang. } \frac{2\acute{\alpha}}{n}\pi = \frac{\overset{a}{\Sigma} P^{\frac{r}{n}} \text{ Sin. } \left(\frac{r}{n} - 2a\right)\varphi}{\overset{a}{\Sigma} P^{\frac{r}{n}} \text{ Cos. } \left(\frac{r}{n} - 2a\right)\varphi}$$

si $(2 \text{ Cos. } \varphi)^r$ positivus est; sin negativus

$$(2 \text{ Cos } \varphi)^{\frac{r}{n}} = \frac{\overset{a}{\Sigma} P^r_{\frac{}{n}} \text{ Cos.} \left(\frac{r}{n} - 2\, a\right) \varphi}{\text{Cos.} \frac{2\, \acute{a} + 1}{n} \pi} (-1)^{\frac{1}{n}}$$

$$= \frac{\overset{a}{\Sigma} P^r_{\frac{}{n}} \text{ Sin.} \left(\frac{r}{n} - 2\, a\right) \varphi}{\text{Cos.} \frac{2\, \acute{a} + 1}{n} \pi} (-1)^{\frac{1}{n}}$$

$$\text{Tang.} \frac{2\, \acute{a} + 1}{n} \pi = \frac{\overset{a}{\Sigma} P^r_{\frac{}{n}} \text{ Sin.} \left(\frac{r}{n} - 2\, a\right) \varphi}{\overset{a}{\Sigma} P^r_{\frac{}{n}} \text{ Cos.} \left(\frac{r}{n} - 2\, a\right) \varphi}$$

$\acute{a}$ adhuc definiendum constituit $= \frac{r}{2}$. Sed formulam, ex qua profectus est, non accuratius excutit nec qualem arcum φ notet, commemorat.

§. 21.

Summam a praecedentibus discrepantem invenit Münchow (Grundlehren der Trigonometrie 1826). Exorditur a $2 \text{ Cos. } \varphi = e^{i\varphi} + e^{-i\varphi} = e^{i\varphi}\left(1 + e^{-2i\varphi}\right)$ aut $= e^{-i\varphi}\left(1 + e^{2i\varphi}\right)$; itaque $2\, e^{i\varphi} \text{ Cos. } \varphi = \left(1 + e^{2i\varphi}\right)$

et $2\, e^{-i\varphi} \text{ Cos. } \varphi = \left(1 + e^{-2i\varphi}\right)$;

quibus ad potestatem m erutis, additione et subtractione invicem sibi additis efficiuntur

$$2^m \left(e^{i\varphi m} + e^{-i\varphi m}\right) \text{Cos. } \varphi^m = 2^m . 2 \text{ Cos. m } \varphi . \text{ Cos. } \varphi^m$$
$$= \left(1 + e^{2i\varphi}\right)^m + \left(1 + e^{-2i\varphi}\right)^m$$
$$= \Sigma P^a_m \left\{e^{2ai\varphi} + e^{-2ai\varphi}\right\} = \Sigma P^a_m . 2 \text{ Cos. } 2\, a\, \varphi.$$

$$2^m \left(e^{i\varphi m} - e^{-i\varphi m}\right) \text{Cos.}\,\varphi^m = 2^m.\,2\,\text{Sin.}\,m\,\varphi.\,\text{Cos.}\,\varphi^m = \left(1+e^{2i\varphi}\right)^m - \left(1+e^{-2i\varphi}\right)^m$$

$$= \Sigma P_m^a \left\{ e^{2ai\varphi} - e^{-2ai\varphi} \right\} = \Sigma P_m^a\, 2\,\text{Sin.}\,2\,a\,\varphi$$

Itaque $|2\ \text{Cos.}\,\varphi|^m\ \text{Cos.}\ m\ \varphi = \Sigma\ P_m^a\,\text{Cos.}\ 2\ a\ \varphi$

et $|2\,\text{Cos.}\,\varphi|^m\ \text{Sin.}\ m\ \varphi = \Sigma\ P_m^a\ \text{Sin.}\ 2\ a\,\varphi.$

Realis potestatis valor invenitur, quia calculus binomialibus coefficientibus factus est.

Arcus φ inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ jacere debet. Sed formulae facile extra hos fines extendi possunt; nam si $\psi = \varphi \pm \alpha\pi$, fit

$$\text{Cos.}\ m\,\psi = \text{Cos.}\ m\ \varphi\ \text{Cos.}\ m\ \alpha\,\pi \mp \text{Sin.}\ m\ \varphi\ \text{Sin.}\ m\ \alpha\ \pi$$
$$\text{Sin.}\ m\,\psi = \text{Sin.}\ m\ \varphi\ \text{Cos.}\ m\ \alpha\ \pi \pm \text{Cos.}\ m\ \varphi\ \text{Sin.}\ m\ \alpha\ \pi$$

ubi pro Cos. m φ et Sin. m φ inventi valores ponendi sunt; porro

$\text{Cos.}\,\varphi = \text{Cos.}\ (\psi \mp \alpha\ \pi) = \text{Cos.}\ \psi\ \text{Cos.}\ \alpha\ \pi.$ et $\text{Cos.}\ 2\ a\ \varphi = \text{Cos.}\ 2\ a\ \psi$ sicuti $\text{Sin.}\ 2\ a\,\varphi = \text{Sin.}\ 2\ a\ \psi$

Itaque $(\text{Cos.}\ \alpha\,\pi)^m\ |2\ \text{Cos.}\ \psi|^m\ \text{Cos.}\ m\ \psi =$

$$\text{Cos.}\ m\ \alpha\,\pi\ \Sigma\ P_m^a\ \text{Cos.}\ 2\ a\ \psi \mp \text{Sin.}\ m\ \alpha\,\pi\,\Sigma\,P_m^a\ \text{Sin.}\ 2\ a\ \psi.$$

$(\text{Cos.}\,\alpha\ \pi)^m\ |2\ \text{Cos.}\ \psi|^m\ \text{Sin.}\ m\ \psi =$

$$\text{Sin.}\ m\ \alpha\ \pi\,\Sigma\ P_m^a\ \text{Cos.}\ 2\ a\ \psi \pm \text{Cos.}\ m\ \alpha\ \pi\,\Sigma\,P_m^a\ \text{Sin.}\ 2\,a\,\psi.$$

quo modo quaeque cosinus potestas secundum cosinus et sinus arcuum multiplicium evolvi potest. Sed prima formula per Cos. m ψ, secunda per Sin. m ψ multiplicata

$(\text{Cos.}\ \alpha\ \pi)^m\ |2\,\text{Cos.}\ \psi|^m\,\text{Cos.}\ m\ \psi^2 =$

$$\text{Cos.}\,m\,\psi\,\text{Cos.}\ m\,\alpha\,\pi\,\Sigma\ P_m^a\ \text{Cos.}\ 2\ a\ \psi \mp \text{Sin.}\,m\,\alpha\,\pi\,\Sigma\ P_m^a\ \text{Sin.}\ 2\,a\ \psi.\ \text{Cos.}\ m\,\psi;$$

$(\text{Cos.}\ \alpha\,\pi)^m\ |2\,\text{Cos.}\ \psi|^m\ \text{Sin.}\,m\ \psi^2 =$

$$\text{Sin.}\,m\,\alpha\ \pi\ \Sigma P_m^a\ \text{Cos.}\ 2\ a\,\psi\ \text{Sin.}\ m\,\psi \pm \text{Cos.}\ m\ \alpha\pi\ \Sigma\ P_m^a\,\text{Sin.}\,2\,a\ \psi\ \text{Sin.}\ m\ \psi$$

additione conjunctae reddunt

$(\text{Cos.}\ \alpha\ \pi)^m |2\ \text{Cos.}\ \psi|^m =$

$$\text{Sin.}\,m\,\alpha\,\pi\,\Sigma\ P_m^a\ \left\{\text{Cos.}\ 2\ a\ \psi\ \text{Sin.}\ m\ \psi \mp \text{Sin.}\ 2\ a\ \psi\ \text{Cos.}\,m\ \psi\right\}$$
$$+\ \text{Cos.}\ m\ \alpha\pi\ \Sigma\ P_m^a \left\{\text{Cos.}\,2\,a\,\psi\ \text{Cos.}\,m\,\psi \pm \text{Sin.}\ 2\,a\,\psi\ \text{Sin.}\ m\ \psi\right\}$$

$$= \text{Sin.}\, m\alpha\pi \Sigma P_m^a \text{Sin.}\, (m-2a)\psi + \text{Cos.}\, m\alpha\pi \Sigma P_m^a \text{Cos.}\, (m-2a)\psi$$

$$(\text{Cos.}\,\alpha\pi)^m |2\, \text{Cos.}\,\psi|^m = \pm \{2\, \text{Cos.}\, (\varphi + \alpha\pi)\}^m =$$

$$\text{Sin.}\, m\alpha\pi \Sigma P_m^a \text{Sin.}\, (m-2a)(\varphi + \alpha\pi).$$

$$+ \text{Cos.}\, m\alpha\pi \Sigma P_m^a \text{Cos.}\, (m-2a)(\varphi + \alpha\pi)$$

ubi $+$ ponendum est, si α par, $-$, si α impar numerus integer est.

§. 22.

Si ad propositae quaestionis summas inventas earumque methodos, quas breviter, sed ita ut auctores eas protulerunt, exponere studebam, respicimus, praecipue commemorandae sunt:

$$(2\, \text{Cos.}\,\varphi)^m = \Sigma P_m^a \text{Cos.}\, (m-2a)(\varphi + 2\alpha\pi)$$
$$+ i \Sigma P_m^a \text{Sin.}\, (m-2a)(\varphi + 2\alpha\pi), \qquad \text{I}$$

quam **Poisson** primus praebuit (cf. § 11)

$$(2\, \text{Cos.}\,\varphi)^m = \frac{(\pm 1)^m}{\text{Cos.}\, m(\text{arc.Cos.} \pm 1)} \Sigma P_m^a \text{Cos.}\, (m-2a)\varphi \qquad \text{II}$$

$$\text{aut } (2\, \text{Cos.}\,\varphi)^m = \frac{(\pm 1)^m}{\text{Cos.}\, m\alpha\pi} \Sigma P_m^a \text{Cos.}\, (m-2a)(\varphi + \alpha\pi) \qquad \text{III}$$

$$= \frac{(\pm 1)^m}{\text{Sin.}\, m\alpha\pi} \Sigma P_m^a \text{Sin.}\, (m-2a)(\varphi + \alpha\pi) \qquad \text{IV}$$

quas formulas **Poinsot** (§ 14 et 15) attulit et quarum casus speciales sunt, quos Poisson § 11 et Cauchy (§ 19) invenerunt. Denique Münchow constituit

$$\pm |2\, \text{Cos.}\,\psi|^m = \text{Sin.}\, m\alpha\pi \Sigma P_m^a \text{Cos.}\, (m-2a)\psi$$
$$+ \text{Cos.}\, m\alpha\pi \Sigma P_m^a \text{Sin.}\, (m-2a)\psi, \qquad \text{V}$$

si ψ jacet inter $(\alpha - \frac{1}{2})\pi$ et $(\alpha + \frac{1}{2})\pi$; et in aequationibus (III et IV) φ arcum inter $\pm \frac{1}{2}\pi$ significat. — Quae expressiones veram summam exhibent, sed methodi, quibus nituntur, non satis probatae sunt, nec causae expositae, cur Euleri formula certis modo finibus circumscripta recta sit; quibuscum cohaeret judicium de argumentationibus (§§ 8 et 9) prolatis

$$o = \Sigma P_m^a \text{Sin.}\, (m-2a)\varphi$$

§. 23.

Quae, ut ex ultimis exordiar, fundatae sunt seriebus

$$\text{Sin.}\,\varphi = \Sigma \frac{(-1)^a \varphi^{2a+1}}{\Pi 2a+1} = \frac{e^{i\varphi} - e^{-i\varphi}}{2}$$

$$\text{Cos.}\,\varphi = \Sigma \frac{(-1)^a \varphi^{2a}}{\Pi 2a} = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

Contendo autem, has series modo valere, si φ arcus minimus est, qui sinui et cosinui respondet. Nam series

$$\text{Sin.}\,\varphi = \varphi - \tfrac{1}{6}\varphi^3 + \frac{\varphi^5}{\Pi 5} - \frac{\varphi^7}{\Pi 7} \ldots . (-1)^a \frac{\varphi^{2a+1}}{\Pi 2a+1} \ldots .$$

$$\text{Cos.}\,\varphi = 1 - \tfrac{1}{2}\varphi + \frac{\varphi^4}{\Pi 4} - \frac{\varphi^6}{\Pi 6} \ldots . (-1)^a \frac{\varphi^{2a}}{\Pi 2a}$$

tantum hoc casu convergunt, de divergentium serierum autem summa sermo esse non potest. Praeterea autem in serie evolvenda de minimo arcu modo agitur. Nam plurimi simili serie ficta primum terminum eo constituunt, ut ostendant, pro minimo arcu Sin. $\varphi = \varphi$, et Cos. $\varphi = 1$. (Brandes polynomischer Lehrsatz pag. 151, Lacroix Traité I pag. 64). — Crelle vero in libro § 17 citato pag. 314 e serie data profectus, demonstraturus est, quomodo ejus summa cum geometrica notione consentiat. Sed quia formula ad omnes curvas quadrante

$$\left(\frac{d\,\text{Cos.}\,y}{dy}\right)^2 + \left(\frac{d\,\text{Sin.}\,y}{dy}\right)^2 = 1$$

utitur, arcum ipsum, id est minimum in circulo, assumit. At postea pag. 329 ad alios quoscunque arcus $y + 2\alpha\pi$ extendere vult, nulla demonstratione adjecta. Quibus probatur, series illas, sinum et cosinum per potestates arcuum exprimentes, modo pro arcubus in primo quadrante valere. Sed $-\varphi$ pro φ posito et Cos. φ et respondens series eundem valorem servant; sed et Sin. φ et series eum exprimens valorem contrario signo afficiendum accipiunt. Quapropter series rectae assumendae sunt, si arcus φ inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ situs est. Ex quo jam elucet, demonstrationes, quas Deflers protulit § 9, modo ad arcus pertinere, qui inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ jaceant.

§. 24.

In illis seriebus locatum est $e^{\pm i\varphi} = \text{Cos.}\, \varphi \pm i\, \text{Sin.}\, \varphi$, quam aequationem itaque ad minimum arcum coercendam censeo. Nam si hoc non esset, deberet $e^{i\varphi} = e^{i(\varphi + 2\alpha\pi)}$, quod etiam Cauchy (cf. § 19) contendit. Sed si imaginariis quantitatibus utimur, semper memores esse debemus, qua notione in calculum inductae sint. Nostro autem casu $e^{i\varphi}$ significat seriem $= \Sigma \frac{(i\varphi)^a}{\Pi a}$. Ut vero $e^{i\varphi} = e^{i(\varphi + 2\alpha\pi)} = e^{i\varphi} . e^{i2\alpha\pi}$, necesse est, ut $e^{i2\alpha\pi} = 1$. Sed

$$e^{i2\alpha\pi} = 1 - \frac{4\,\alpha^2\pi^2}{2} + \frac{16\,\alpha^4\pi^4}{\Pi 4} - \frac{64\,\alpha^6\pi^6}{\Pi 6} + \frac{(2\,\alpha\,\pi)^8}{\Pi 8}$$

$$- \frac{(2\,\alpha\,\pi)^{10}}{\Pi 10} \ldots\ldots + (-1)^a \frac{(2\,\alpha\,\pi)^{2a}}{\Pi 2a} \ldots\ldots$$

$$+ i \left\{ 2\,\alpha\,\pi - \frac{(2\,\alpha\,\pi)^3}{\Pi 3} + \frac{(2\,\alpha\,\pi)^5}{\Pi 5} - \frac{(2\,\alpha\,\pi)^7}{\Pi 7} + \right.$$

$$\left. \frac{(2\,\alpha\,\pi)^9}{\Pi 9} \ldots\ldots (-1)^a \frac{(2\,\alpha\,\pi)^{2a+1}}{\Pi 2a+1} \ldots\ldots \right\}$$

Desideratur ergo, ut utraque series pro se evanescat; quia vero α quemcunque numerum integrum significare potest, utraque series divergit, neque ullam summam habet, neque $=$ o fit. Quam ob causam $e^{i\varphi}$ non $= e^{i(\varphi + 2\alpha\pi)}$ ponendum est, nisi aliis argumentis probatum.

§. 25.

Ex $e^{i\varphi} = \text{Cos.}\, \varphi + i\, \text{Sin.}\, \varphi$ sequitur etiam theorema, quod Moiveri nomen fert, nempe

$$e^{im\varphi} = (\text{Cos.}\, \varphi + i\, \text{Sin.}\, \varphi)^m = \text{Cos.}\, m\,\varphi + i\, \text{Sin.}\, m\,\varphi,$$

quod itaque valere tantum potest, si φ inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ situm est. Si vero ψ quicunque arcus est, exprimi potest $\psi = \varphi \pm \alpha\pi$ et

$$|\text{Cos.}\,\psi \pm i\,\text{Sin.}\,\psi|^m = \frac{\text{Cos.}\,m\psi \pm i\,\text{Sin.}\,m\psi}{\text{Cos.}\,m\alpha\pi \pm i\,\text{Sin.}\,m\alpha\pi},$$

quod sequenti modo demonstrari potest.

E formulis $e^{2\varphi i} = \frac{1 + i\,\text{tang.}\,\varphi}{1 - i\,\text{tang.}\,\varphi}$, et $e^{-2\varphi i} = \frac{1 - i\,\text{tang.}\,\varphi}{1 + i\,\text{tang.}\,\varphi}$

deducitur $\left(1 + e^{2\varphi i}\right)^m \pm \left(1 + e^{-2\varphi i}\right)^m = 2^m \frac{(1 + i\,\text{tang.}\,\varphi)^m \pm (1 - i\,\text{tang.}\,\varphi)^m}{(1 + \text{tang.}\,\varphi^2)^m}$

et $\left(1 + e^{2\varphi i}\right)^m + \left(1 + e^{-2\varphi i}\right)^m = e^{m\varphi i}\left(e^{\varphi i} + e^{-\varphi i}\right)^m + e^{-i\varphi m}\left(e^{\varphi i} + e^{-\varphi i}\right)^m =$

$$\left(e^{\varphi i} + e^{-\varphi i}\right)^m \left\{e^{m\varphi i} + e^{-m\varphi i}\right\} = |2\,\text{Cos.}\,\varphi|^m\, 2\,\text{Cos.}\,m\,\varphi.$$

Ergo $2.\,2^m \text{Cos.}\,m\varphi\,\text{Cos.}\,\varphi^m = 2^m \frac{(1 + i\,\text{tang.}\,\varphi)^m + (1 - i\,\text{tang.}\,\varphi)^m}{(\text{Sec.}\,\varphi^2)^m}$

$$2\,\text{Cos.}\,m\,\varphi = \text{Cos.}\,\varphi^m\, \Sigma\,(-1)^a\, 2\, P_m^{2a}\, \text{tang.}\,\varphi^{2a}$$

$$\text{Cos.}\,m\,\varphi = \Sigma\,(-1)^a\, P_m^{2a}\, \text{Sin.}\,\varphi^{2a}\, \text{Cos.}\,\varphi^{m-2a};$$

simili ratione $\text{Sin.}\,\varphi = \Sigma\,(-1)^a\, P_m^{2a+1}\, \text{Sin.}\,\varphi^{2a+1}\, \text{Cos.}\,\varphi^{m-(2a+1)}$

ubi φ semper inter fines $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ sumendum est.

$\text{Cos.}\,m\,\varphi + i\,\text{Sin.}\,m\,\varphi = |\text{Cos.}\,\varphi + i\,\text{Sin.}\,\varphi|^m.$ Sed

$$\text{Cos.}\,m\,\psi = \text{Cos.}\,m\,\varphi\,\text{Cos.}\,m\,\alpha\pi - \text{Sin.}\,m\,\varphi\,\text{Sin.}\,m\,\alpha\pi$$

$$i\,\text{Sin.}\,m\,\psi = i\,\text{Cos.}\,m\,\varphi\,\text{Sin.}\,m\,\alpha\pi + i\,\text{Sin.}\,m\,\varphi\,\text{Cos.}\,m\,\alpha\pi$$

$$\text{Cos.}\,m\,\psi + i\,\text{Sin.}\,m\,\psi = (\text{Cos.}\,m\,\varphi + i\,\text{Sin.}\,m\,\varphi)\,(\text{Cos.}\,m\,\alpha\pi + i\,\text{Sin.}\,m\,\alpha\pi)$$

Sed $\text{Cos.}\,m\,\varphi + i\,\text{Sin.}\,m\,\varphi = |\text{Cos.}\,\varphi + i\,\text{Sin.}\,\varphi|^m$, quia calculus per binomiales coefficientes institutus est. Hic quidem de reali valore sermo esse non potest, quia radix quantitatem imaginariam complectitur, sed valorem significamus, qui eodem modo ex radice oritur, quo reales valores exsistunt. Quod etiam ita ostendi potest:

$$(\text{Cos.}\,\varphi + i\,\text{Sin.}\,\varphi)^m = \left(e^{i\varphi}\right)^m = \Sigma \left(\frac{(i\varphi)^a}{\Pi a.}\right)^m$$

Si $e^{i\varphi m}$ in seriem explico et potestatem radicis $\Sigma\, \frac{(i\,\varphi)^a}{\Pi\, a}$ per binomiales coefficientes evolvo, ad eandem summam pervenio. Sed valor binomialibus coefficientibus evolutus realis est, aut certe ei respondet.

Est vero $(\text{Cos.}\ \varphi + i\ \text{Sin.}\ \varphi)^m =$

$$(e^{i\varphi})^m = \left\{1 + i\varphi - \tfrac{1}{2}\varphi^2 - \tfrac{1}{6}\varphi^3 i + \tfrac{1}{24}\varphi^4 + \tfrac{1}{120}\varphi^5 i - \tfrac{1}{720}\varphi^6 \ldots\right\}^m$$

$$\begin{array}{lllllll}
= 1 + i\,m\varphi & -m\frac{\varphi^2}{2} & -\frac{m\varphi^3 i}{6} & +\frac{m\,\varphi^4}{24} & +\frac{m\varphi^5 i}{120} & -\frac{m\varphi^6}{720} & \\
 & -P_m^2\,\varphi^2 & -2P_m^2\frac{i\varphi^3}{2} & +P_m^2\frac{\varphi^4}{4} & +2P_m^2\frac{i\varphi^5}{24} & -P_m^2\frac{\varphi^6}{36} & \\
 & & & +2P_m^2\frac{\varphi^4}{6} & +2P_m^2\frac{i\varphi^5}{12} & -2P_m^2\frac{\varphi^6}{120} & \\
 & & & & & -2P_m^2\frac{\varphi^6}{48} & \\
 & & -P_m^3\,i\,\varphi^3 & +3P_m^3\frac{\varphi^4}{2} & +3P_m^3\frac{i\varphi^5}{6} & -P_m^3\frac{\varphi^6}{8} & \\
 & & & & +3P_m^3\frac{i\varphi^5}{4} & -3P_m^3\frac{\varphi^6}{24} & -6P_m^3\frac{\varphi^6}{12} \\
 & & & +P_m^4\,\varphi^4 & +4P_m^4\frac{i\varphi^5}{2} & -4P_m^4\frac{\varphi^6}{6} & -6P_m^4\frac{\varphi^6}{4} \\
 & & & & +P_m^5\,i\varphi^5 & -5P_m^5\frac{\varphi^6}{2} & \\
 & & & & & -6P_m^6\,\varphi^6 &
\end{array}$$

Coefficientibus cujusque potestatis in unum contractis accipimus seriem

$$1 + i\,m\varphi - \frac{(m\ \varphi)^2}{2} - \frac{i\,(m\ \varphi)^3}{6} + \frac{(m\ \varphi)^4}{24} + \frac{i\,(m\,\varphi)^5}{120} - \frac{(m\ \varphi)^6}{720}\ \ldots$$

$$= e^{i\,m\varphi} = \text{Cos.}\ m\ \varphi + i\ \text{Sin.}\ m\ \varphi.$$

Ut vero, si m numerus fractus est $= \frac{r}{n}$, omnes potestatis valores accipiamus, illum per unitatis radices multiplicamus.

$$(\text{Cos.}\,\varphi + i\,\text{Sin.}\,\varphi)^m = |\text{Cos.}\,\varphi + i\,\text{Sin.}\,\varphi|^m (+1)^m$$
$$= (\text{Cos.}\,m\,\varphi + i\,\text{Sin.}\,m\,\varphi)(\text{Cos.}\,2\,m\,\alpha\,\pi \pm \text{Sin.}\,2\,m\,\alpha\,\pi)$$
$$= \text{Cos.}\,m\,(\varphi + 2\,\alpha\,\pi) \pm i\,\text{Sin.}\,m\,(\varphi + 2\,\alpha\,\pi)$$

ubi α quemcunque numerum integrum significare potest, sed modo ponendi sunt pro eo numeri a zero usque ad $-\frac{1}{4}$ r et $+\frac{1}{4}$ r, ne $\frac{2\alpha n}{r}\pi$ praescriptos fines $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ transgrediatur. Ex quibus in promptu est, quae expressio adhibenda sit, si de reali solo valore sermo est, et qua expressione uti debeamus, ut omnes valores exhibeamus.

§. 26.

Quum itaque Euler his inter certos tantum fines valentibus formulis utatur, summa ab eo inventa his tantum finibus coercita recta esse potest. Sed formula illa etiam negativorum cosinuum potestates exprimi possunt; nam si $\psi = \varphi + (2\,\alpha + 1)\,\pi$, ergo $\text{Cos.}\,\varphi = -\text{Cos.}\,\varphi$, tunc est $(2\,\text{Cos.}\,\psi)^m = (-2\,\text{Cos.}\,\varphi)^m = |2\,\text{Cos.}\,\varphi|^m\,(-1)^m = (-1)^m\,\Sigma\,P_m^a\,\text{Cos.}\,(m-2\,a)\,\varphi$, ubi vero in serie alius arcus est, ac in radice evolvenda.

Sequitur etiam e demonstratione § 8 exposita $\Sigma\,P_m^a\,\text{Sin.}\,(m-2\,a)\,\varphi = 0$ semper valere, si φ inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$.

§. 27.

Praeterea monitum est, Euleri formulam non rectam esse, quia unum tantum valorem exprimeret; quod vero ei non vituperationi, sed laudi est. Nam Euler valorem realem quaesivit, quod methodi, quibus usus est, demonstrant, nempe per binomiales coefficientes potestatem evolvit et $(\text{Cos.}\,\varphi \pm i\,\text{Sin.}\,\varphi)^m = \text{Cos.}\,m\,\varphi \pm i\,\text{Sin.}\,m\,\varphi$ posuit. Itaque formulae (1 et 2 § 3) non duo varii potestatis valores esse possunt, ut Poisson loco § 5 citato contendit. Illi duo oriuntur ex $(y+z)^m$ et $(z+y)^m$, itaque e quantitatibus aliter positis; varii modo esse possunt, si altera series aut utraque divergit, quia de talis seriei valore nihil constituere possumus. Nostro autem casu convergunt series. — Nam $\Sigma\,P_m^a\,x^a$ (quam seriem Abel — Crelle Journal I No. 29 — pertractat et calculo per generales functiones instituto, etiam praepositae nostrae quaestionis solutionem cum (III et IV § 22) consentientem invenit) convergit, si m positive et x inter -1 et $+1$ sumptum est.

Nam quotus duorum terminorum invicem se excipientium $= \frac{m - a}{a + 1} x$ indicatis conditionibus numerus vere fractus est. Nostro autem casu idem coefficientes quantitatibus inter $+ 1$ et $- 1$ jacentibus multiplicantur, quae vero non semper decrescunt, sed periodis mutantur. Exemplum § 7 allatum ostendit, non semper sequentem terminum antecedente minorem esse, quia majore sinu aut cosinu coefficientem superante multiplicatur, verbi causa $P_{\frac{7}{9}}^{14}$ Sin. $85 > P_{\frac{7}{9}}^{13}$ Sin. 31. Nihilo minus nostrae series convergunt, quia idem sinus et cosinus redeunt minoribus coefficientibus multiplicandi. Quam ob causam convergentibus illis seriebus recte concludi potest, non varios valores exhiberi, sed

$$|2 \text{ Cos. } \varphi|^m = \Sigma P_m^a \text{ Cos. } (m - 2a)\varphi \pm i \Sigma P_m^a \text{ Sin. } (m - 2a)\varphi$$
$$= \Sigma P_m^a \text{ Cos. } (m - 2a)\varphi$$

si m positiva quantitas et φ arcus inter $- \frac{1}{2}\pi$ et $+ \frac{1}{2}\pi$ est. Ad negativos cosinus Euleri formula non quadrat, quia negativae quantitatis potestates per coefficients binomiales bene exprimi non possunt, quia x in $(1 + x)^m = \Sigma P_m^a x^a$ minor quam $- 1$ esset. Sed

$$|2 \text{ Cos. } \varphi|^{-m} = \frac{1}{|2 \text{ Cos. } \varphi|^{+m}} = \frac{1}{\Sigma P_m^a \text{ Cos. } (m - 2a)\varphi}.$$

§. 28.

Quum Poisson (§ 11)

$$(\text{Cos. } \varphi + i \text{ Sin. } \varphi)^m = \text{Cos. } m(\varphi + \alpha\pi) \pm i \text{ Sin. } m(\varphi + \alpha\pi)$$

adhibeat, summa, ad quam perducitur, universe omnes valores praebere oportet. Sed deinde § 12 φ melius inter $- \frac{1}{2}\pi$ et $+ \frac{1}{2}\pi$ assumpsisset; nam si φ inter o et π jacet, $\text{Cos. } \vartheta = \frac{(1 + y) \text{ Cos. } \varphi}{R}$ signum mutat; nam Cos. ϑ positivus esse debet, si φ in primo, negativus, si φ in secundo quadrante jacet, quia tunc Cos. φ negativa quantitas est, R autem semper absolutum valorem significare debet. Quod Poisson praetermisit. — Omnino illa limitatio ad plurimas prolatarum methodorum pertinet, quae calculis, ea subjectis, utuntur. Ita summa, quam Ohm affert (§ 16)

$$(2\,\mathrm{Cos.}\,\varphi)^m = \mathrm{Cos.}\,2\,\alpha\,m\,\pi \sum P_m^a\,\mathrm{Cos.}\,m - 2\,a)\,\varphi$$
$$- \mathrm{Sin.}\,2\,m\,\alpha\,\pi \sum P_m^a\,\mathrm{Sin.}\,(m - 2\,a)\,\varphi +$$
$$i \left\{ \mathrm{Sin.}\,2\,m\,\alpha\,\pi \sum P_m^a\,\mathrm{Cos.}\,(m - 2\,a)\,\varphi + \mathrm{Cos.}\,2\,m\,\alpha\,\pi \sum P_m^a\,\mathrm{Sin.}\,(m - 2\,a\,\varphi) \right\}$$

tantummodo forma a recta discrepat, quia terminos nihili aequales, nempe $\sum P_m^a\,\mathrm{Sin.}\,(m - 2\,a)\,\varphi = o$ continet; quibus neglectis fit

$$(2\,\mathrm{Cos.}\,\varphi)^m = \sum P_m^a\,\mathrm{Cos.}\,(m - 2\,a)\,\varphi \left\{ \mathrm{Cos.}\,2\,\alpha\,m\,\pi + i\,\mathrm{Sin.}\,2\,\alpha\,m\,\pi \right\}.$$

Ohm quidem bene principia exposuit, quae in formulis plures valores exprimentibus observanda sint; sed, ut mihi videtur, quum formulae non semper omnes valores notent, quos significare possunt, methodorum, quibus ad eum perventum sit, ratio habenda est.

§. 29.

At calculus, quem Lagrange secutus est, illis formulis non nititur et Poinsot ostendit, quomodo ea veram summam invenire possimus. Ohm quidem contradicit, indefinitorum coefficientium methodum modo adhibendam esse, si antea probatum sit, functionem tali serie evolvi posse. Sed puto, hoc ex ipso calculo processurum esse, si series secundum potestates variabilis progreditur, ubi calculus ita institui potest, ut, summa seriei $= o$ facta, praeter unam omnes potestates evanescant, itaque illius potestatis coefficientes invicem se tollere debeant. — Quod autem in nostra serie locum non habet, quia idem a variabili dependens valor saepius recurrere potest, itaque extra coefficientes in calculum vocatos etiam alii respiciendi essent. Quam ob causam censeo, seriem ita inventam non necessario rectam, sed alio modo adhuc probandam esse. Poinsot autem attulit

$$\left\{ 2\,\mathrm{Cos.}\,(\varphi + \alpha\,\pi) \right\}^m = \frac{(\pm 1)^m}{\mathrm{Cos.}\,m\,(\mathrm{arc.\,Cos.} \pm 1)} \sum P_m^a\,\mathrm{Cos.}(m - 2\,a)\,(\varphi + \alpha\pi),$$

$$\text{aut} = \frac{(\pm 1)^m}{\mathrm{Sin.}\,m\,(\mathrm{arc.\,Cos.} \pm 1)} \sum P_m^a\,\mathrm{Sin.}\,(m - 2\,a)\,(\varphi + \alpha\,\pi).$$

Quod probatur summa, quam Münchow (§ 21) praebet

$$\pm \left| 2\,\mathrm{Cos.}\,(\varphi + \alpha\pi) \right|^m = \mathrm{Sin.}\,m\,\alpha\,\pi \sum P_m^a\,\mathrm{Sin.}\,(m - 2\,a)\,(\varphi + \alpha\,\pi)$$
$$+ \mathrm{Cos.}\,m\,\alpha\,\pi \sum P_m^a\,\mathrm{Cos.}\,(m - 2\,a)\,(\varphi + \alpha\,\pi).$$

calculo accuratissimo usus, quia formulas (§ 23 — 26) commemoratas semper inter praescriptos fines admovet. Forma quidem haec summa discrepat, sed Poinsot ostendit (§ 14)

$$\frac{\Sigma P_m^a \text{Cos.} (m - 2a)(\varphi + \alpha\pi)}{\Sigma P_m^a \text{Sin.} (m - 2a)(\varphi + \alpha\pi)} = \frac{\text{Cos. } m\alpha\pi}{\text{Sin. } m\alpha\pi}.$$

Itaque si pro ΣP_m^a Cos. $(m - 2a)(\varphi + \alpha\pi)$ in illa serie ponimus Cotang. $m\alpha\pi\, \Sigma P_m^a$ Sin. $(m - 2a)(\varphi + \alpha\pi)$, efficitur $|2 \text{ Cos.} (\varphi + \alpha\pi)|^m$ $= \Sigma P_m^a \text{Sin.} (m - 2a)(\varphi + \alpha\pi) \left\{ \text{Cotang. } m\alpha\pi) \text{ Cos. } m\alpha\pi + \text{Sin. } m\alpha\pi \right\}$

$$= \frac{1}{\text{Sin. } m\alpha\pi} \Sigma P_m^a \text{Sin.} (m - 2a)(\varphi + \alpha\pi)$$

quia Cotang. $m\alpha\pi$ Cos. $m\alpha\pi$ + Sin. $m\alpha\pi$

$$= \frac{\text{Cos. } m\alpha\pi^2}{\text{Sin. } m\alpha\pi} + \text{Sin. } m\alpha\pi = \frac{\text{Cos. } m\alpha\pi^2 + \text{Sin. } m\alpha\pi^2}{\text{Sin. } m\alpha\pi}$$

§. 30.

Sed variationem constantis quantitatis A (§ 14) Poinsot non bene probat. Dicit enim formulam inventam n. n. valores habere non posse, sed modo n; et tali ratione denominatorem Cos. m (arc. Cos. $\pm$ 1) constituit. Quod autem indicere possit, formulam non rectam esse. Itaque argumento utitur, quod adhuc probandum est. Sed quia series notatae pro arcubus inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ valent, deinde autem ad arcus plures peripherias majores aut minores extendi possunt, denominator ille cum arcu, fines transgresso, mutandus est.

§. 31.

Ex quibus omnibus sequitur, formulam rectam, omnes valores exprimentem, esse

$$\left\{ 2 \text{ Cos.} (\varphi + \alpha\pi) \right\}^m = \frac{(\alpha\pi)^m}{\text{Cos. } m\alpha\pi} \Sigma P_m^a \text{Cos.} (m - 2a)(\varphi + \alpha\pi)$$

$$\text{aut} = \frac{(\alpha\pi)^m}{\text{Sin. } m\alpha\pi} \Sigma P_m^a \text{Sin.} (m - 2a)(\varphi + \alpha\pi),$$

ubi per φ arcus inter $-\frac{1}{2}\pi$ et $+\frac{1}{2}\pi$ significatur, α autem quemcunque numerum integrum positivum aut negativum exprimere potest.

Quum vero Sin. φ = Cos. $(\frac{1}{2}\pi - \varphi)$ sit, ergo $(2 \text{ Sin. } \varphi)^m = \left\{2 \text{ Cos. } (\frac{1}{2}\pi - \varphi)\right\}^m$ in formulis illis, ut potestatem sinus exhibeamus, pro φ, $\frac{1}{2}\pi - \varphi$ ponendum est. Itaque

$$\left\{2 \text{ Cos. } (\tfrac{1}{2}\pi - \varphi + \alpha\pi)\right\}^m = \left\{2 \text{ Sin. } (\varphi + \alpha\pi)\right\}^m =$$

$$\frac{(\alpha\pi)^m}{\text{Cos. } m\alpha\pi} \Sigma P_m^a \text{ Cos. } (m - 2a) \left\{(\tfrac{1}{2} + \alpha)\pi - \varphi\right\}$$

$$\text{aut} = \frac{(\alpha\pi)^m}{\text{Sin. } m\alpha\pi} \Sigma P_m^a \text{ Sin. } (m - 2a) \left\{(\tfrac{1}{2} + \alpha)\pi - \varphi\right\}$$

§. 32.

Restat adhuc de unitatis radicibus nonnulla addere, de quibus Crelle et Olivier (§§ 17, 18) egerunt. Postulant in disquisitionibus, ut exponens fractus minimis numeris expressus sit; quod autem fieri non potest, nisi potestatis valore mutato. Nam exempli gratia $a^{\frac{7}{21}}$ non poni potest $= a^{\frac{1}{3}}$, quia $a^{\frac{7}{21}} = \sqrt[21]{a^7}$ unum et viginti valores habet, sed $\sqrt[3]{a}$ tres tantummodo exhibet; ex quo jam ambarum expressionum discrepantia elucet. Addamus $(-1)^{\frac{2}{4}} = \sqrt[4]{(-1)^2} = \sqrt[4]{+1} = \pm 1$ aut $\pm \sqrt{-1}$; sed $(-1)^{\frac{1}{2}}$ modo est $= \pm\sqrt{-1}$. Itaque videmus, valores potestatis $(-1)^{\frac{1}{2}}$ contineri in $(-1)^{\frac{2}{4}}$, quod autem praeterea adhuc alios valores habet.

Quamobrem casus, quos Crelle affert, modo valent, si exponens datus jam minimis numeris exprimitur. — Ut vero omnes casus comprehendamus, melius est, quantitatem radicalem ad eam potestatem evolvere, quam nominator exponentis notat, et ex hac potestate radicem ejus gradus, quem denominator significat, extrahere. Quod ad unitatis potestates applicemus.

Si exponentis nominator par est, $(\pm 1)^{\frac{2r}{n}} = \sqrt[n]{(\pm 1)^{2r}}$ fit $= \sqrt[n]{+1}$, cui semper valor realis respondet $= \pm 1$, si n etiam numerus par est, sed modo $= + 1$, si n impar. Valores omnino imaginarii exstant, si n numerus quadripartitus est. Si vero exponentis nominator impar est, sejungenda

sunt $(+1)^{\frac{2r+1}{n}}$ et $(-1)^{\frac{2r+1}{n}}$. Nam $(+1)^{\frac{2r+1}{n}} = \sqrt[n]{1^{2r+1}} = \sqrt[n]{+1}$, ubi pro n eaedem conditiones ac supra valent; sed $(-1)^{\frac{2r+1}{n}} = \sqrt[n]{(-1)^{2r+1}} = \sqrt[n]{-1}$, ubi realis valor exstat, si n numerus impar est; ut imaginarius valor locum habeat, n formae $4\beta + 2$ esse debet.

Pagani (§ 20) recte e radice constituenda exit et potestatis valores bene explicat, sed ad summas certo rectas pervenire non potest, quia aequationem, ad quam formulas suas applicat, omisso examine omnino valentem assumit, nec usquam dicit, qualem arcum per φ significet. Sed rectas formulas exhibet pro cosinibus, quorum arcus in primo aut secundo quadrante siti sunt.

Vita.

Franciscus Christophorus Felix Bredow natus sum VI ante Calendas Octobris 1803 Eutini patre tunc scholae in illa urbe florentis rectore, deinde Helmstadii, Francofurti ad Viadrum, denique in hac alma Universitate historiae tradendae provincia functo, matre e gente Weisseriana. Quum vero pater, jam Vratislaviam vocatus, antea Berolinum se contulisset, viro sibi amicissimo professori Plamann, qui scholae secundum Pestalozziana principia instituta praeerat, me tradidit, ut prima litterarum initia discerem. Patre autem Vratislaviae 1815 mortuo, vir ille liberalissimus cum uxore sua benevolentissima parentum instar me curabant et in quacunque vitae conditione tot beneficia in me conferebant, ut nunquam eas, quas debeo, gratias iis persolvere possim. Tirociniis litterarum absolutis in gymnasii Berolinensis tertiam classem receptus, doctissimorum praeceptorum institutionibus gavisus sum, e quibus praecipue directori Bellermann, professori Koepke, nunc illius gymnasii directori, et professori E. G. Fischer, qui ad matheseos studium me incitavit posteaque benigno consilio adjuvit, optime de me meritis, quam maximam gratiam habeo semperque habebo. Anno 1824 e gymnasio cum testimonio primae notae dimissus Vratislaviam me contuli, ubi a rectore illustrissimo Wendt numero civium academicorum adscriptus, exegeticis, philosophicis, philologicis, historicis, mathematiciset physicis lectionibus virorum pro ordine unius cujusque venerandorum Brandes, Branis, Fischer, Jungnitz, Passow, Schneider, Schulz, Steffens, Stenzel, Thilo, Wachler interfui, quibus operibus quam verbis ostendere mallem,

eos non sine fructu beneficia in me cumulasse. Perspicuitas, qua Brandes mathemathicas disciplinas tractavit, fervorque, quo Steffens physica tradidit, ita me incitaverunt, ut totum his artibus me darem. Postea inde a festo paschali anni 1826 Halae mathematica studia persequebar, duce professore Scherk, qui et illo anno et omni postea tempore quam humanissime mihi favebat et ut praemissam dissertationem scriberem, auctor mihi fuit. Duorum autem virorum, quibus multa debeo, mortem lugeo, Ersch et Reisig, quorum hic festiva orationis vi et ingenii acumine florens eximie antiquitatis indolem exponere et ad studium ipsum excitare scivit. Praeterea Halae Gesenium de archaeologia judaica, Meyer de Graecis philosophis, Hoffmann de montibus ignivomis disputantes audivi.

Triennio academico absoluto Berolinum abii et examine pro venia in gymnasiis docendi probatus in Friderico-Guilelmiano gymnasio arithmeticae et geometriae elementa per unum annum tradidi. Non possum, quin eximiam humanitatem viri maxime reverendi Enke commemorem, qui instrumenta astronomica cognoscendi provinciam mihi dedit et practicae astronomiae principia mihi exposuit. Exeunte mense Septembri 1828 Vratislaviam iterum me contuli, ubi singulari benignitate professoris Middeldorpf seminarii paedagogici sodalis factus in gymnasio Elisabethaneo de matheseos et physices elementis praecepi. Qua data occasione benevolum rectoris Reiche in me animum merita laude celebrare omittere nequeo. A festo paschali in Olsnensi gymnasio, quod directore Koerner floret, praeceptoris munus suscepi.